Combination Resonances of a Circular Plate With Three-Mode Interaction

Won Kyoung Lee
Associate Professor,
Mem. ASME,
Department of Mechanical Engineering,

Cheol Hong Kim
Researcher,
Institute of Industrial Technology,
Yeungnam University,
Gyongsan 712-749, Korea

1 Introduction

A clamped circular plate experiences midplane stretching when deflected. The influence of this stretching on the dynamic response increases with the amplitude of the response. This situation can be described with nonlinear strain-displacement equations and a linear stress-strain law which gives us the nonlinear partial differential equation. Since the linear analysis for this problem gives us reasonable results for only small motions, we need the nonlinear analysis for large motions. Even though a circular plate is a continuous system, it can be approximated as a two-degree-of-freedom system. When a nonlinear multi-degree-of-freedom system has two or more of its natural frequencies commensurable or nearly so, the system may possess internal resonances (which can lead to strong modal interactions).

Many authors have examined responses of multi-degree-of-freedom systems with internal resonances to study nonlinear dynamic behaviors of structural elements (beam and plate) with fixed boundaries. Nayfeh et al. (1974) considered two-mode interactions to study the steady-state responses of a hinged-clamped beam under harmonic excitation. They found that the system can have multiple stable steady-state responses when certain resonance conditions are satisfied. In this case, Lee and Ghang (1994) obtained domains of attraction of the beam to recognize the set of initial states attracting eventually to each stable response. Lee and Soh (1994) included three-mode interaction for the hinged-clamped beam problem to compare two-mode and three-mode interactions. They found no significant difference between both modal interactions. It means that the more modes included in the analysis don’t guarantee any different result for the problem. Nayfeh and his colleagues (Sridhar et al., 1975, 1978; Nayfeh and Mook, 1979; Hadian and Nayfeh, 1990) studied the primary resonance of a clamped circular plate with internal resonance and recently found the occurrence of Hopf bifurcations. Nayfeh and Mook (1979) studied a combination resonance of the hinged-clamped beam, and found the nonexistence of resonance responses and the decrease of resonance amplitudes with the excitation amplitude. It seems that they didn’t pay much attention to these peculiar phenomena because they were interested in the resonance responses only.

In this paper, combination resonances in the symmetric responses of the clamped circular plate with a three-mode interaction is studied. We are interested in the resonance responses of the generalized coordinates as well as the actual responses of the plate. We numerically integrate the nonlinear nonautonomous ordinary differential equations governing the time-dependent coefficients in Galerkin’s procedure to find the long-term behavior of the plate and to check the validity of the analytical solution. In order to visualize total responses depending on the initial conditions, we draw the deflection curves of the plate.

2 Equations of Motion and Steady-State Responses

For convenience, we refer to the existing works (Nayfeh et al., 1975; Nayfeh and Mook, 1979; Hadian and Nayfeh, 1990) for the equations of motion for the generalized coordinates as follows:

\[ \frac{d^2\psi_n}{dt^2} + \omega_n^2\psi_n = \epsilon \left( -2c_n \frac{d\psi_n}{dt} + \sum_{m} \sum_{p,q} \Gamma_{mqpq} \psi_m \psi_p \psi_q \right) + f_s(t) \]  

where \( \epsilon = (12(1 - \nu^2))h^3/\pi^2 = R^2 \) dimensionless small parameter; \( \nu = \) Poisson ratio; \( h = \) thickness of the plate; \( R = \) radius of the plate; \( c_n = \) damping ratio; \( \Gamma_{mqpq} = \) nonlinear coefficients; and \( f_s(t) = 2K_n \cos \Omega t \) is the excitation (see the Appendix, Part
Substituting Eqs. (7) and (8) into Eq. (1) and equating the coefficients of equal powers of \(e\), we obtain

\[e^k; D_0^k \psi_{n0} + \omega_0^2 \psi_{n0} = 2K_e \cos \Omega T_e, \quad n = 1, 2, \ldots (9)\]

\[e^1; D_0^1 \psi_{n1} + \omega_1^2 \psi_{n1} = -2D_0 D_1 \psi_{n0} - 2c_D D_0 \psi_{n0} + \sum_{m,p,q=1}^{27} \Gamma_{mnpq} \psi_m \psi_p \psi_q \psi_{n0}, \quad n = 1, 2, \ldots (10)\]

\[e^2; \quad \psi_{n0} = A_n (T_1, T_2, \ldots) \exp(i \omega_n T_0) + \Lambda_n \exp(i \Omega T_0) + cc\]

\[n = 1, 2, \ldots (11)\]

where \(K_e\) is assumed to be \(O(1)\). It is convenient to write the solution to (9) as follows:

\[\psi_{n0} = A_n (T_1, T_2, \ldots) \exp(i \omega_n T_0) + \Lambda_n \exp(i \Omega T_0) + cc\]

\[n = 1, 2, \ldots (11)\]

where \(A_n = K_e/(\omega_0^2 - \Omega^2)\) and cc represents the complex conjugate of the terms to the left. The \(A_n\) are unknown at this point, but they are determined from the elimination of secular terms at the next level of approximation. Substituting (11) into (10) leads to

\[D_0^2 \psi_{n1} + \omega_1^2 \psi_{n1} = -2i \omega_0 (D_1 A_n + c_D A_n) \exp(i \omega_n T_0) - 2i \omega_1 A_0 \exp(i \Omega T_0) + \sum_{m,p,q=1}^{27} \Gamma_{mnpq} \psi_m \psi_p \psi_q \psi_{n0}\]

\[n = 1, 2, \ldots (12)\]

where \(\Gamma_{mnpq}\) and \(\lambda_j\) are given in the Appendix, Part 1.

In order to eliminate the secular terms from the \(\psi_{n1}\), the \(A_n\) must be chosen so that the coefficient of \(\exp(i \omega_n T_0)\) is zero. This coefficient will contain the nonlinear terms when the following conditions are satisfied:

\[\omega_n \approx \omega_m \pm \omega_p \pm \omega_q. (13)\]

We can see that the condition (5) is one of the cases (13) with \(n = 3, m = 1, p = q = 2\) and all signs of plus.

Substituting the resonance conditions (5) and (6) into Eqs. (12), and eliminating the secular terms, we have the following solvability conditions:

\[n = 1:\]

\[-2i \omega_1 (A_1 + c_A A_1) + A_1 \sum_{j=1}^{n} \alpha_j A_j \bar{A}_j + Q_1 A_1 \exp(-i \sigma_1 T_1) + 2H_1 A_1 + 2F_1 A_1 \exp(i \sigma_1 T_1) + GA_1 \exp(i \eta_1 (\sigma_1 + \sigma_2) T_1) = 0 (14a)\]

\[n = 2:\]

\[-2i \omega_2 (A_2 + c_A A_2) + A_2 \sum_{j=1}^{n} \alpha_j A_j \bar{A}_j + Q_2 A_1 A_2 \exp(-i \sigma_1 T_1) + 2H_2 A_2 + FA_1 A_2 \exp(i \sigma_1 T_1) + GA_1 A_2 \exp(-i (\sigma_1 + \sigma_2) T_1) = 0 (14b)\]

\[n = 3:\]

\[-2i \omega_3 (A_3 + c_A A_3) + A_3 \sum_{j=1}^{n} \alpha_j A_j \bar{A}_j + Q_3 A_1 A_2 \exp(i \sigma_1 T_1) + 2H_3 A_3 + GA_1 A_2 \exp(i (\sigma_1 + \sigma_2) T_1) = 0 (14c)\]

\[n = 4:\]

\[-2i \omega_4 (A_4 + c_A A_4) + A_4 \sum_{j=1}^{n} \alpha_j A_j \bar{A}_j + 2H_4 A_1 A_2 = 0 (14d)\]
where

\[ \alpha_{nj} = \begin{cases} 4\Gamma_{nj} + 2\Gamma_{nj}^* & (n \neq j) \\ 3\Gamma_{nn} & (n = j) \end{cases} \]

\[ Q_1 = (2\Gamma_{123} + \Gamma_{132}) = Q_3 \]

\[ Q_2 = 2Q_1 \]

\[ H_{ak} = \sum_{m,j=1}^N (2\Gamma_{akm} + \Gamma_{akj})a_n a_j \]

\[ F = \sum_{j=1}^N (2\Gamma_{1j} + \Gamma_{1j})a_j \]

\[ G = 2 \sum_{j=1}^N (\Gamma_{123} + \Gamma_{132} + \Gamma_{122})a_j \]

and the prime denotes the derivative with respect to \( T_1 = \epsilon t \). The values of the coefficients are given in the Appendix, Parts 4 and 6. The coefficients \( Q_1 \) and \( H_{ak} \) are due to the internal resonance and the secondary resonance (nonprimary resonance), respectively. The coefficients \( F \) and \( G \) are due to the combination resonances. The coefficients \( \alpha_{nj} \) are not relevant to these resonances and appear even in the primary resonance and nonresonance cases.

To solve Eqs. (14), we write \( A_n \) in the polar form

\[ A_n = \frac{1}{2}a_n \exp(i\beta_n), \quad n = 1, 2, \ldots \]

where \( a_n \) and \( \beta_n \) are real. Then we separate the result into real and imaginary parts and obtain

\[ n = 1, 2, 3: \]

\[ \omega_1 (a_1^2 + c_1 a_1) + Q_3 a_3^2 a_3 \sin \mu_1 \]

\[ + \frac{1}{2} F a_1 a_2 \sin \mu_2 + \frac{1}{2} G a_3 a_3 \sin (\mu_2 - \mu_1) = 0 \] (15a)

\[ \omega_1 a_1 \beta_1' + \frac{1}{2} \sum_{j=1}^N a_1 a_j^2 + Q_3 a_3 a_3 \cos \mu_1 + H_1 a_1 a_1 \]

\[ + \frac{1}{2} F a_2 a_2 \cos \mu_2 + \frac{1}{2} G a_3 a_3 \cos (\mu_2 - \mu_1) = 0 \] (15b)

\[ \omega_2 (a_2^2 + c_2 a_2) + 2Q a_3 a_3 \sin \mu_1 \]

\[ + \frac{1}{2} F a_1 a_2 \sin \mu_2 - \frac{1}{2} G a_3 a_3 \sin (\mu_2 - \mu_1) = 0 \] (15c)

\[ \omega_2 a_2 \beta_2' + \frac{1}{2} \sum_{j=1}^N a_2 a_j^2 + 2Q a_3 a_3 \cos \mu_1 + H_2 a_2 a_2 \]

\[ + \frac{1}{2} F a_3 a_3 \cos \mu_2 + \frac{1}{2} G a_3 a_3 \cos (\mu_2 - \mu_1) = 0 \] (15d)

\[ \omega_3 (a_3^2 + c_3 a_3) - Q_3 a_3^2 a_3 \sin \mu_1 \]

\[ + \frac{1}{2} G a_3 a_3 \sin (\mu_2 - \mu_1) = 0 \] (15e)

\[ \omega_3 a_3 \beta_3' + \frac{1}{2} \sum_{j=1}^N a_3 a_j^2 + Q_3 a_3 a_3 \cos \mu_1 + H_3 a_3 a_3 \]

\[ + \frac{1}{2} G a_3 a_3 \cos (\mu_2 - \mu_1) = 0 \] (15f)

\[ n = 4: \]

\[ \omega_4 (a_4^2 + c_4 a_4) = 0 \] (16a)

\[ \omega_4 a_4 \beta_4' + \frac{1}{2} \sum_{j=1}^N a_4 a_j^2 + H_4 a_4 a_4 = 0 \] (16b)

where

\[ 8Q = Q_i \]

\[ \mu_1 = \beta_1 + 2\beta_2 - \beta_3 + \sigma_1 T_1 \]

\[ \mu_2 = 2\beta_1 + \beta_2 - \sigma_2 T_1. \] (17a)

From Eq. (16a) we can see that the amplitudes \( a_n \) \((n \geq 4)\) decay because

\[ a_n \approx \exp(-c_n T_1), \quad n \geq 4. \]

Since we are interested in the steady-state response, we can disregard \( a_n \) \((n \geq 4)\).

Then Eqs. (15) can be reduced to a set of autonomous ordinary differential equations in amplitudes \( a_n \) and phases \( \mu_n \) for \( n = 1, 2, 3 \) as follows:

\[ a_1' = -c_1 a_1 - \frac{Q_3 a_3}{\omega_1} \sin \mu_1 \]

\[ - \frac{F a_2 a_2}{2\omega_1} \sin \mu_2 - \frac{G a_3 a_3}{4\omega_1} \sin (\mu_2 - \mu_1) \] (18a)

\[ a_2' = -c_2 a_2 - \frac{2Q a_3 a_3}{\omega_2} \sin \mu_1 \]

\[ - \frac{F a_3 a_3^2}{2\omega_2} \sin \mu_2 + \frac{G a_3 a_3}{4\omega_2} \sin (\mu_2 - \mu_1) \] (18b)

\[ a_3' = -c_3 a_3 + \frac{Q a_3 a_3^2}{\omega_3} \sin \mu_1 - \frac{G a_3 a_3}{4\omega_3} \sin (\mu_2 - \mu_1) \] (18c)

\[ a_1 a_2 a_3 \mu_1 = a_1 a_2 a_3 \mu_1 + \frac{a_3 a_3}{8} \sum_{n=1}^3 \left( \frac{a_3}{\omega_n} - \frac{2a_3}{\omega_1} - \frac{a_3}{\omega_3} \right) a_n^2 \]

\[ + \left( \frac{a_3^2}{\omega_3} - \frac{4a_3^2}{\omega_1^2} + \frac{a_3^2}{\omega_1} \right) \cos \mu_1 \]

\[ + \frac{a_3 a_3}{2\omega_3} \left( \frac{2H_{23}}{\omega_2} - \frac{H_{11}}{\omega_2} \right) \cos (\mu_2 - \mu_1) \] (18d)

\[ a_1 a_2 a_3 \mu_2 = -a_1 a_2 a_3 - \frac{a_3}{8} \sum_{n=1}^3 \left( \frac{2a_3}{\omega_1} + \frac{a_3}{\omega_2} \right) a_n^2 \]

\[ - 2Q a_3 a_3 \left( \frac{\alpha_1^2}{\omega_1} + \frac{\alpha_3^2}{\omega_2} \right) \cos \mu_1 - a_3 \left( \frac{2H_{12}}{\omega_1} + \frac{H_{23}}{\omega_2} \right) \]

\[ - F_{\alpha_1} \left( \frac{\alpha_1^2}{\omega_1} + \frac{\alpha_3^2}{2\omega_2} \right) \cos \mu_2 \]

\[ - \frac{G a_3}{\omega_1} \left( \frac{\alpha_1^2}{\omega_1} + \frac{\alpha_3^2}{\omega_2} \right) \cos (\mu_2 - \mu_1). \] (18e)

By the observation of the above equations we can see that there are two types of steady-state responses \((a_1' = a_2' = a_3' = \mu_1 = \mu_2 = 0)\) such as (I) \( a_1 = 0, a_2 = 0, a_3 = 0 \) and (II) \( a_1 = a_2 = a_3 = 0 \). Dividing (18d) and (18e) by \( a_1 a_2 a_3 \) and \( a_2 a_3 \), respectively, we can obtain the following algebraic equations giving the steady-state response of \( a_1 \neq 0, a_2 \neq 0, a_3 \neq 0 \).

\[ \omega_1 a_1 + \frac{Q a_3 a_3}{2} \sin \mu_1 + \frac{F a_3 a_3}{2} \sin \mu_2 \]

\[ + \frac{G a_3 a_3}{4} \sin (\mu_2 - \mu_1) = 0 \] (19a)

\[ \omega_2 a_2 + 2Q a_3 a_3 \sin \mu_1 + \frac{F a_3 a_3}{2} \sin \mu_2 \]

\[ + \frac{G a_3 a_3}{4} \sin (\mu_2 - \mu_1) = 0 \] (19b)
3 Asymptotic Stability Analysis

3.1 Steady-State Resonance Responses When $a_1 = 0$, $a_2 = 0$, $a_3 = 0$. Let one of the steady-state resonance responses of $a_1 = 0$, $a_2 = 0$, $a_3 = 0$ be $a_{10}$, $a_{20}$, $a_{30}$, $\mu_{10}$, $\mu_{20}$. To determine the stability of the various steady-state responses we examine the system behavior for small disturbance on the steady state. Let

$$a_1 = a_{10} + \delta a_1, \quad a_2 = a_{20} + \delta a_2, \quad a_3 = a_{30} + \delta a_3,$$

$$\mu_1 = \mu_{10} + \delta \mu_1, \quad \mu_2 = \mu_{20} + \delta \mu_2 \quad (26)$$

where $\delta(\cdot)$ represents a small disturbance of the steady state. Substituting Eq. (26) into Eq. (18), and retaining only linear terms leads to

$$d \frac{d}{dT} \begin{bmatrix} \delta a_1 \\ \delta a_2 \\ \delta a_3 \\ \delta \mu_1 \\ \delta \mu_2 \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} \\
\Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} \\
\Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} & \Phi_{35} \\
\Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & \Phi_{45} \\
\Phi_{51} & \Phi_{52} & \Phi_{53} & \Phi_{54} & \Phi_{55} \end{bmatrix} \begin{bmatrix} \delta a_1 \\ \delta a_2 \\ \delta a_3 \\ \delta \mu_1 \\ \delta \mu_2 \end{bmatrix}. \quad (27)$$

The steady-state response is considered asymptotically stable if the real parts of all eigenvalues of the coefficient matrix $[\Phi]$ are negative.

3.2 Steady-State Resonance Response When $a_1 = a_2 = a_3 = 0$. In this case $a_{10} = a_{20} = a_{30} = 0$ but $\mu_{10}$ and $\mu_{20}$ are indeterminate. In other words, this solution is a singular solution; thus, for the stability of the singular solution we cannot rely on the Eq. (27).

In order to use Eq. (14) instead of Eq. (27), we introduce a small disturbance as follows:

$$A_n = A_{n0} + \delta A_n, \quad n = 1, 2, 3 \quad (28)$$

where

$$A_{n0} = \frac{1}{2} a_{n0} \exp(i \beta_{n0}) = 0, \quad n = 1, 2, 3. \quad (29)$$

Thus

$$A_n = \delta A_n, \quad n = 1, 2, 3. \quad (30)$$

Substituting Eq. (30) into Eq. (14) we have

$$\delta A_n' = - \left( c_n + i \frac{H_{nn}}{\omega_n} \right) \delta A_n, \quad n = 1, 2, 3. \quad (31)$$

Let

$$\delta A_n = \delta A_{nr} + i \delta A_{ni}, \quad n = 1, 2, 3 \quad (32)$$

where $\delta A_{nr}$ and $\delta A_{ni}$ are real.

Substituting Eq. (32) into Eq. (31) we have

$$d \frac{d}{dT} \begin{bmatrix} \delta A_{nr} \\ \delta A_{ni} \end{bmatrix} = \begin{bmatrix} -c_n & \frac{H_{nn}}{\omega_n} \\
- \frac{H_{nn}}{\omega_n} & -c_n \end{bmatrix} \begin{bmatrix} \delta A_{nr} \\ \delta A_{ni} \end{bmatrix}. \quad (33)$$

The eigenvalues of the coefficient matrix in Eq. (33) are

$$\lambda_{1,2} = -c_n \pm i \frac{H_{nn}}{\omega_n}, \quad n = 1, 2, 3. \quad (34)$$

Therefore the real parts of all eigenvalues are $-c_n < 0$ because the damping coefficients are positive. Conclusively the zero-amplitude response turns out to be stable as long as the damping coefficients are positive.

4 Numerical Results

Using Eq. (19) and stability criteria mentioned earlier, we have plotted the amplitude-parameter $(c_{sr} = \delta_2$ and $B$) response
curves shown in Figs. 2 and 3, where solid and dotted lines denote, respectively, stable and unstable responses. There exist two pairs of nonzero-amplitude responses. Each pair has the stable and unstable branches. Since the zero-amplitude response is stable, the system has at most three stable responses. In this case, the long-term response of the system depends on the initial condition.

For convenience, let say, the first two stable responses are the nonzero-amplitude responses and the third stable response is the zero-amplitude response. Then the amplitude \(a_{nj}(n, j = 1, 2, 3)\) denoted by solid lines implies the amplitude of the \(n\)th mode of the \(j\)th stable response. Thus we have \(a_{1j} \neq 0, a_{2j} \neq 0, a_{3j} \neq 0, a_{12} \neq 0, a_{23} \neq 0, a_{33} \neq 0, a_{13} = 0, a_{23} = 0, a_{33} = 0\). According to this notation, we need to modify Eqs. (20), (23), and (24) as follows:

\[
\psi_{nj} = \begin{cases} 
 a_{nj} \cos (\omega t + \beta_{nj}) + 2\Lambda_n \cos \Omega t + O(\varepsilon), & n = 1, 2, 3 \\
 2\Lambda_n \cos \Omega t + O(\varepsilon), & n \equiv 4
\end{cases} \tag{20'}
\]

where

\[
w_j(r, t) = w_{crj}(r, t) + w_{cr}(r, t) \tag{23'}
\]

In these expressions, the generalized coordinates \(\psi_{nj}\), the total deflection \(w_j\) and the combination-resonance deflection \(w_{crj}\) are corresponding to the \(j\)th stable response. In order to check the validity of the analytical results, we have integrated numerically the system (1) and used the Eq. (20') to obtain the amplitudes of the stable steady-state responses. The symbols \(\square, \bigcirc, \text{ and } \Delta\) denote the first, second, and third stable responses, respectively.

In Fig. 2, the amplitudes of three modes are shown as functions of \(\delta_2\). We have bifurcations at \(\delta_{2A}\) and \(\delta_{2B}\). When \(\delta_2 < \delta_{2A}\), the only stable response is the zero-amplitude response, \(a_{3j}\). When \(\delta_{2A} < \delta_2 < \delta_{2B}\), there exist two stable responses, \(a_{1j}\) and \(a_{2j}\). When \(\delta_2 > \delta_{2B}\), there exist three stable responses, \(a_{1j}, a_{2j}\), and \(a_{3j}\). Since the amplitude \(a_{3j}\) is almost zero, the amplitude and zero-amplitude the response \(a_{3j}\) almost overlap with each other in the figure. Because of the existence of multiple stable responses, we have two jumps at \(\delta_{2A}\) and \(\delta_{2B}\). By means of the numerical integration of the system (1), it has been found that both stable responses jump down to the zero-amplitude response as \(\delta_2\) decreases.

In Fig. 3 the amplitudes are plotted as functions of \(B\). The amplitudes of the both nonzero-amplitude responses, \(a_{1j}\) and \(a_{2j}\), decrease with the excitation amplitude. This phenomenon is very peculiar even in nonlinear systems, though Nayfeh and Mook (1979) had also found this phenomenon in the combination resonance of a hinged-clamped beam. We have bifurcations at \(B_{A1}, B_{B9}, B_{C}, \text{ and } B_D\). When \(B < B_A\) and \(B > B_B\), the only stable response is the zero-amplitude response, \(a_{3j}\). When \(B_A < B < B_B\), there exist two stable responses, \(a_{1j}\) and \(a_{2j}\). When \(B_A < B < B_C\), there exist three stable responses, \(a_{1j}, a_{2j}, a_{3j}\). When \(B_C < B < B_D\), there exist two stable responses, \(a_{1j}\), and \(a_{3j}\). By means of the numerical integration, jump phenomena are found at \(B_A, B_B, B_C, \text{ and } B_D\). All jumps but at \(B_B\) are downward to \(a_{3j}\). At \(B_B\) the response \(a_{2j}\) jumps to \(a_{3j}\). The jump at \(B_B\) cannot be recognized as a downward jump because the second mode jumps up, while the first and third modes jump down.

The variation of nonresonance response \(\Lambda_1\) with the detuning parameter \(\varepsilon\gamma\) obtained from Eq. (22) is shown in Fig. 4 where \(\Lambda_1\) and \(\Lambda_2\) decrease and \(\Lambda_3\) increases with \(\varepsilon\gamma\). This results from the fact that the excitation frequency \(\Omega\) is greater than \(\omega_1\) and \(\omega_2\), and is less than \(\omega_3\). In Fig. 5 the variation of nonresonance responses with the excitation amplitude \(B\) is shown. The proportionality of \(\Lambda_3\) to \(B\) tells us that those nonresonance responses are just the responses of the linearized system. This proportionality of linear responses to \(B\) is in contrast to the decrease of amplitudes of nonzero-amplitude responses (or nonlinear responses) with \(B\).

Integrating the nonautonomous system (1) and using Eq. (20'), we can draw Figs. 6–8 showing the time histories of three stable steady-state responses. We have used values for the system parameters as the following:
\{ \hat{\delta}_2, \hat{c}_n, B, \epsilon \} = \{ 1.0, 0.01, 26500, 1.063 \times 10^{-3} \}. \tag{34}

Since the system has three asymptotically stable steady-state responses, the long-term behavior of the plate depends on the initial condition. The initial conditions used for the figures are as follows:

\[
\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \]

= \{ 7.14, 0.0, 4.27, 0.0, 3.36, 0.0 \} \quad \text{for Fig. 6}

\[
\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \]

= \{ -0.40, 0.0, 4.76, 0.0, 1.29, 0.0 \} \quad \text{for Fig. 7}

\[
\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \]

= \{ -3.65, 0.0, 3.66, 0.0, 2.27, 0.0 \} \quad \text{for Fig. 8}

For the numerical integration, in this study, we employ the Runge-Kutta fourth-order method with the time step size of integration, \( \Delta t = 2\pi/(30\omega_0) \). The results in Figs. 6(b), 7(b), and 8(b) have been used to construct the numerical integration results in Figs. 2 and 3.

Using the analytical results and Eqs. (20)', (23)', and (24)', we have plotted the nonresonance deflections \( \psi_m(r, t) \), the combination resonance deflections \( \psi_m(r, t) \), and the total deflections \( w_j(r, t) \) in the steady states, shown in Fig. 9. By means of Eqs. (17) and (21), the constant phases \( \gamma_{aj} \) are chosen as the follows:

\[
\gamma_{aj} = 0, \quad \gamma_{2j} = \gamma_{3j} = 2\gamma_{2j} - \gamma_{1j}
\]

where \( \gamma_{1j} \) and \( \gamma_{2j} \) are the steady-state values of the \( j \)-th stable responses obtained analytically. The values of system parameters used for this figure are the same as ones in Eq. (34). The results in the figure are for the duration \( 2T \), where \( T \) is the excitation period \( 2\pi/\Omega \). We need to note that the nonresonance deflection \( \psi_m \) has the same period as the excitation period \( T \), because it is a linear solution. The initial conditions determine which deflection of three deflections \( w_j \) represents the actual deflection.

In order to show how the parameter \( B \) and \( \hat{\delta}_2 \) influence the system response, we plot stability regions in \( B - \hat{\delta}_2 \) plane shown in Fig. 10. Bifurcations occur on two curves. In region I, there exist one stable response, \( a_{n3} \). In region II, there exist two stable responses, \( a_{n2} \) and \( a_{n3} \). In region III, there exist three stable responses, \( a_{n1}, a_{n2}, \) and \( a_{n3} \). In region IV, there exist two stable responses, \( a_{n1} \) and \( a_{n3} \).
Acknowledgment

This work was supported in part by the Korea Science and Engineering Foundation under Grant KOSEF 89-0202-03 and by the Korea Research Foundation under Grant Non-directed Research Fund 1993. The authors gratefully acknowledge one of the referees for his particularly helpful comments on this paper.

References


APPENDIX

1. Coefficients $S_j$ and $\lambda_j$ in Eq. (12):

$$\sum_{j=1}^{27} S_j \exp(i\lambda_j T_0).$$

<table>
<thead>
<tr>
<th>$S_j$</th>
<th>$\lambda_j$</th>
<th>$S_j$</th>
<th>$\lambda_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{12} \lambda_2 \omega_1 - \omega_2 \omega_1 \omega_3 \omega_4 \omega_5 \omega_6 \omega_7 \omega_8 \omega_9 \omega_{10} \omega_{11} \omega_{12} \omega_{13} \omega_{14} \omega_{15} \omega_{16} \omega_{17} \omega_{18} \omega_{19} \omega_{20} \omega_{21} \omega_{22} \omega_{23} \omega_{24} \omega_{25} \omega_{26} \omega_{27}$</td>
<td>$\omega_1 \omega_2 \omega_3 \omega_4 \omega_5 \omega_6 \omega_7 \omega_8 \omega_9 \omega_{10} \omega_{11} \omega_{12} \omega_{13} \omega_{14} \omega_{15} \omega_{16} \omega_{17} \omega_{18} \omega_{19} \omega_{20} \omega_{21} \omega_{22} \omega_{23} \omega_{24} \omega_{25} \omega_{26} \omega_{27}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Nonlinear coefficients $\Gamma_{nmpq}$:

$$\Gamma_{nmpq} = \sum_{k=1}^{\infty} \frac{\varphi_k \varphi^*_m \varphi^*_n \varphi^*_q (\zeta_0 r) dr \varphi_k \varphi^*_m \varphi^*_n \varphi^*_q (\zeta_0 r) dr}{(\zeta_0^2 - 1 + \nu^2) f_i (\zeta_0)}.$$

where

$J_0$: the Bessel function of the first kind of order 0
$J_1$: the Bessel function of the first kind of order 1
$I_0$: the modified Bessel function of the first kind of order 0
$k_m = \nu \omega_m$

Journal of Applied Mechanics
\( \zeta_n \) are roots of the equation

\[ \zeta J_0(\zeta) - (1 + \nu)J_1(\zeta) = 0. \]

3 Coefficients \( c_n \) and \( f_n \):

\[ c_n = \int_0^1 r \varphi_n dr \]

\[ f_n(t) = \int_0^1 r \varphi_n f(r, t) dr. \]

4 Coefficients in Eq. (14):

\[ \alpha_{11} = 3 \Gamma_{1111}, \quad \alpha_{12} = 2(2 \Gamma_{1212} + \Gamma_{1122}) = \alpha_{21}, \]

\[ \alpha_{23} = 2(2 \Gamma_{1313} + \Gamma_{1123}) = \alpha_{31}, \]

\[ \alpha_{22} = 3 \Gamma_{2222}, \quad \alpha_{23} = 2(2 \Gamma_{2323} + \Gamma_{2332}) = \alpha_{32}, \quad \alpha_{33} = 3 \Gamma_{3333}, \]

5 For uniformly distributed loads:

\[ 2K_s = B \int_0^1 r \varphi_n(r) dr. \]

6 Coefficients \( H_{nk} \), \( F \), and \( G \):

\[ H_{nk} = \sum_{m=1}^{3} \sum_{j=1}^{3} (2 \Gamma_{mnkj} + \Gamma_{mnkj}) \Lambda_m \Lambda_j, \]

\[ F = \sum_{j=1}^{3} (2 \Gamma_{12j} + \Gamma_{11j}) \Lambda_j, \]

\[ G = 2 \sum_{j=1}^{3} (\Gamma_{13j} + \Gamma_{13j} + \Gamma_{13j}) \Lambda_j. \]