Second-order approximation for chaotic responses of a harmonically excited spring–pendulum system

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Abstract

The influence of a higher-order approximation on chaotic responses of a weakly non-linear multi-degree-of-freedom system is investigated. The specific system examined is a harmonically excited spring–pendulum system, which is known to be a good model for a variety of engineering systems, including ship motions. By the method of multiple scales the original system is reduced to a second-order approximate system. The long-term behaviors of both systems are compared by examining the largest Lyapunov exponents. It is observed that the second-order approximation gives better qualitative agreement with the original system than the first-order approximation does. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

After chaotic motions known to occur only in strongly non-linear systems had been found in weakly non-linear multi-degree-of-freedom systems [1–3], many researchers have investigated chaotic motions of weakly non-linear multi-degree-of-freedom systems [4–9]. In their works, transforming the original systems into approximate systems and examining the approximate systems, they have found Hopf bifurcations and a sequence of period-doubling bifurcations leading to chaos. They conjectured that chaos in the approximate systems implies chaos in the original systems, but there is no theorem stating for the relationship between chaos of the approximate system and solutions of the original system. In this light, Bajaj and his colleagues have made extensive investigations to show that there seems to be a qualitative correspondence between chaotic solutions of the approximate system and those of the original system [10–14]. Lee and Park [15] added some more evidences to this correspondence by comparing the largest Lyapunov exponents for long-term responses of both systems.

Meanwhile, higher-order expansions have been used to obtain better predictions for responses of
single- [16, 17] and multi- [18, 19] degree-of-freedom systems, though Banerjee et al. [20] found that the second-order averaging can give physically unrealizable results. In the previous work [15], we found that in some cases the first-order expansion is unsatisfactory in predicting the largest Lyapunov exponent. Thus, in this paper we investigate the influence of a higher-order expansion on the chaotic responses for a harmonically excited spring–pendulum system with internal resonance. The resonance conditions considered are as follows: internal resonance, $\omega_1 \approx 2\omega_2$ and external resonance, $\Omega = \omega_2$, where $\omega_1$, $\omega_2$ and $\Omega$ denote natural frequencies of the spring and pendulum modes and the forcing frequency, respectively. Using the method of multiple scales, we transform the original system into a second-order approximate system. The long-term behaviors of both systems are compared by examining the largest Lyapunov exponents.

2. Equations of motion and analysis

For convenience, we refer to the existing works [15, 21, 22] on a harmonically excited spring–pendulum system in Fig. 1 for the equations of motion for the normal coordinates as follows:

$$\ddot{x} + c_1 \dot{x} + \omega_1^2 x - (1 + x)\phi^2 + \omega_2^2(1 - \cos \varphi) = f_1 \cos \Omega t,$$

$$(1 + x)^2 \dot{\varphi} + c_2 \dot{\varphi} + 2(1 + x)\dot{x}\dot{\varphi} + \omega_2^2(1 + x)\sin \varphi = f_2 \cos \Omega t,$$

where $x$ is the non-dimensional coordinate of the spring motion, $\varphi$ the coordinate of the pendulum motion, $\omega_1$ the natural frequency of the spring mode, $\omega_2$ the natural frequency of the pendulum mode, $c_1$, $c_2$ the damping coefficients, $f_1$, $f_2$ the forcing amplitudes and $\Omega$ the forcing frequency.

Motions in some small neighborhood of the static equilibrium position are considered so that the amplitude of the response is assumed to be of the order of a small parameter $\varepsilon$, $0 < \varepsilon \ll 1$. The linear viscous damping forces and the exciting forces are assumed to be of the order of $\varepsilon^2$ such that

$$c_n = \varepsilon^2 \hat{c}_n, \quad f_n = \varepsilon^2 \hat{f}_n, \quad n = 1, 2.$$  \hspace{1cm} (2)

The parameters $\omega_1$, $\omega_2$, $\hat{\omega}_1$, $\hat{\omega}_2$, $\hat{f}_1$, $\hat{f}_2$ and $\Omega$ are of the order of 1.

The behavior of such a system can be very complex, especially when the natural frequencies and the forcing frequency satisfy certain internal and external resonance conditions. Here, we assume that $\omega_1 \approx 2\omega_2$ and $\Omega \approx \omega_2$. To describe how close the frequencies are to the resonance conditions we introduce detuning parameters:

$$2\omega_2 = \omega_1 + \sigma_1 = \omega_1 + \varepsilon \hat{\sigma}_1,$$

$$\Omega = \omega_2 + \sigma_2 = \omega_2 + \varepsilon \hat{\sigma}_2,$$  \hspace{1cm} (3)

where $\sigma_1$ and $\sigma_2$ are called the internal and external detuning parameters, respectively, and $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are of the order of 1. The method of multiple scales [23] is used to obtain a uniformly valid, asymptotic expansion of the solution for Eq. (1) in the case of $2\omega_2 \approx \omega_1$ and $\Omega \approx \omega_2$. For the second-order approximation, introducing three time scales,

$$T_0 = t, \quad T_1 = \varepsilon t \quad T_2 = \varepsilon^2 t.$$  \hspace{1cm} (4)

Fig. 1. A harmonically excited spring–pendulum system.
we seek an asymptotic solution of Eq. (1) for small but finite $x$ and $\varphi$ in the form of

$$x(t; \varepsilon) = \sum_{n=1}^{3} e^n x_n(T_0, T_1, T_2) + O(\varepsilon^4),$$

$$\varphi(t; \varepsilon) = \sum_{n=1}^{3} e^n \varphi_n(T_0, T_1, T_2) + O(\varepsilon^4).$$

(5)

Derivatives with respect to $t$ are then transformed into

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2,$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2),$$

where

$$D_0 = \frac{\partial}{\partial t} T_0, \quad D_1 = \frac{\partial}{\partial t} T_1, \quad D_2 = \frac{\partial}{\partial t} T_2.$$  

(7)

Terms of $O(\varepsilon^3)$ and the higher order in Eq. (6) have been neglected. Substituting Eqs. (4)–(6) into Eq. (1) and equating coefficients of equal power of $\varepsilon$ lead to

$$O(\varepsilon):$$

$$D_0^2 x_1 + \omega_1^2 x_1 = 0,$$

$$D_0^2 \varphi_1 + \omega_1^2 \varphi_1 = 0,$$

$$O(\varepsilon^2):$$

$$D_0^2 x_2 + \omega_1^2 x_2 = -2D_0 D_1 x_1 + (D_0 \varphi_1)^2 - \hat{\gamma}_1 D_0 x_1 - \omega_1^2 \varphi_1^2/2 + \hat{f}_1 \cos \Omega T_0,$$

$$D_0^2 \varphi_2 + \omega_1^2 \varphi_2 = -2D_0 D_1 \varphi_1 - 2D_0 x_1 D_0 \varphi_1 - \hat{\gamma}_2 D_0 \varphi_1 - \omega_1^2 x_1 \varphi_1 - 2x_1 D_0^2 \varphi_1 + \hat{f}_2 \cos \Omega T_0,$$

$$O(\varepsilon^3):$$

$$D_0^2 x_3 + \omega_1^2 x_3 = -2D_0 D_1 x_2 - (D_1^2 + 2D_0 D_2) x_1 - \hat{\gamma}_1 D_0 x_2 - \hat{\gamma}_1 D_1 x_1 + 2D_0 \varphi_1 D_0 \varphi_2 + 2D_0 \varphi_1 D_1 \varphi_1 + x_1 (D_0 \varphi_1)^2 - \omega_1^2 \varphi_1 \varphi_2,$$

$$D_0^2 \varphi_3 + \omega_1^2 \varphi_3 = -2D_0 D_1 \varphi_2 - (D_1^2 + 2D_0 D_2) \varphi_1 - 2x_1 D_0^2 \varphi_2 - 4x_1 D_0 D_1 \varphi_1 - x_1^2 D_0^2 \varphi_1 - 2x_2 D_0^2 \varphi_1 - \hat{\gamma}_2 D_0 \varphi_2$$

$$- \hat{\gamma}_2 D_0 \varphi_3 + \omega_1^2 \varphi_1 \varphi_2.$$  

(10)

The general solution of Eq. (8) can be written in the form

$$x_1 = A_1(T_1, T_2) \exp(i \omega_1 T_0) + c.c.,$$

$$\varphi_1 = A_2(T_1, T_2) \exp(i \omega_2 T_0) + c.c.,$$  

(11)

where c.c. represents the complex conjugate of the previous terms.

Substituting Eq. (11) into Eq. (9) and using the resonance conditions (3) lead to secular terms. Eliminating these secular terms leads to the solvability conditions for the first-order expansion:

$$2i \omega_1 D_1 A_1 = -i \omega_1 \hat{\gamma}_1 A_1 - \frac{3}{2} \omega_2^2 A_2^2 \exp(i \hat{\sigma}_1 T_1),$$

$$2i \omega_2 D_1 A_2 = -i \omega_2 \hat{\gamma}_2 A_2 - \omega_2 (2 \omega_1 - \omega_2) A_1 \bar{A}_2$$

$$\times \exp(-i \hat{\sigma}_1 T_1) + \frac{\hat{f}_2}{2} \exp(i \hat{\sigma}_2 T_1).$$  

(12)

After eliminating the secular terms, the non-homogeneous solutions of Eq. (9) are

$$x_2 = \frac{\hat{f}_1}{2(\omega_1^2 - \Omega^2)} \exp(i \Omega T_0) + \frac{\omega_2^2}{2 \omega_1} A_2 \bar{A}_2 + c.c.,$$

$$\varphi_2 = -\frac{\omega_2 (2 \omega_1 + \omega_2)}{\omega_1 (2 \omega_2 + \omega_1)} A_1 A_2 \exp[i(\omega_1 + \omega_2) T_0] + c.c.$$  

(13)

Substituting Eqs. (11) and (13) into Eq. (10) and using the resonance conditions (3) lead to secular terms in this order. Eliminating these secular terms lead to the solvability conditions for the second-order expansion

$$2i \omega_1 D_2 A_1$$

$$= \frac{\hat{\gamma}_1^2}{4} A_1 + i \frac{\omega_2}{8 \omega_1} (6 \hat{\gamma}_2 \omega_2 - 8 \hat{\gamma}_2 \omega_1$$

$$- 3 \hat{\gamma}_1 \omega_2) A_2^2 \exp(i \hat{\sigma}_1 T_1)$$
\[
2\omega_2 D_2 A_2 = \frac{c_2^2}{4} A_2
\]

\[
= \frac{(12\omega_1^4 - 31\omega_1^3\omega_2 + 18\omega_1\omega_2^3 + 4\omega_2^4)}{4\omega_1(\omega_1 + 2\omega_2)} \times A_1 \tilde{A}_1 A_2 + i \frac{1}{2}(2\hat{c}_1 \omega_1 + 4\hat{c}_2 \omega_1 - 5\hat{c}_1 \omega_2 - 8\hat{c}_2 \omega_2) A_1 \tilde{A}_2 \exp(-i\hat{\sigma}_1 T_1)
\]

\[
- \frac{1}{2} \hat{\sigma}_2(2\omega_1 - \omega_2) A_1 \tilde{A}_2 \exp(-i\hat{\sigma}_1 T_1)
\]

\[
+ \frac{2\omega_2^4(10\omega_1^2 - 15\omega_1 \omega_2 + 8\omega_2^3)}{8\omega_1^2} \times \tilde{A}_2
\]

\[
+ \frac{(\hat{c}_2 - 2\hat{\sigma}_2)}{8\omega_2} f_2 \exp(i\hat{\sigma}_2 T_1)
\]

\[
+ \frac{3(2\omega_1 - 3\omega_2)}{8\omega_2} f_2 A_1 \times \exp[-i(\hat{\sigma}_1 + \hat{\sigma}_2) T_1].
\]

Using Eq. (6) we can express the derivative of \(A_n\) with respect to \(t\) as follows:

\[
2\omega_n \frac{dA_n}{dt} = \varepsilon 2i\omega_n D_1 A_n + \varepsilon^2 2i\omega_n D_2 A_n + O(\varepsilon^3),
\]

\[n = 1, 2.\]  \hspace{1cm} (15)

Let

\[
a_1 = \varepsilon \hat{a}_1, \quad a_2 = \varepsilon \hat{a}_2,
\]

where \(\hat{a}_1\) and \(\hat{a}_2\) are of the order of 1. For convenience, we introduce polar coordinates \(\hat{a}_n, \gamma_n\) are rectangular coordinates \(\hat{p}_n\) for the complex amplitudes as follows:

\[
A_1 = \frac{1}{2}(\hat{p}_1 - i\hat{p}_2) \exp[i(2\hat{\sigma}_2 + \hat{\sigma}_1) T_1],
\]

\[
A_2 = \frac{1}{2}(\hat{p}_3 - i\hat{p}_4) \exp[i\hat{\sigma}_2 T_1],
\]

(17)

where

\[
\hat{p}_1 = \hat{a}_1 \cos(2\gamma_1 + \gamma_2), \quad \hat{p}_2 = \hat{a}_1 \sin(2\gamma_1 + \gamma_2),
\]

\[
\hat{p}_3 = \hat{a}_2 \cos \gamma_1, \quad \hat{p}_4 = \hat{a}_2 \sin \gamma_1.
\]

If we let the rectangular coordinates \(p_n = i\hat{p}_n\), then \(\hat{p}_n\) are of the order of 1. Substituting Eqs. (12) and (14) into Eq. (15) and using Eq. (17) leads to the following system of autonomous ordinary differential equations.

\[
\dot{\hat{p}}_1 = -\frac{1}{2} c_1 p_1 - (2\sigma_2 + \sigma_1) p_2 + \frac{3\omega_2^2}{4\omega_1} p_3 p_4
\]

\[
+ \left\{ -\frac{c_1^2}{8\omega_1} p_2 - \frac{3\sigma_1 \omega_2^2}{8\omega_1^2} p_3 p_4 + \frac{(3\omega_2 - 4\omega_1) f_2}{16\omega_1^2} p_4
\]

\[
- \frac{\omega_2(8c_2 \omega_1 + 3c_1 \omega_2 - 6c_2 \omega_2)}{32\omega_1^2}(p_3^2 - p_4^2)
\]

\[
+ \frac{\omega_2(8\omega_1^3 + 14\omega_1^2 \omega_2 - 17\omega_1 \omega_2^2 + 10\omega_2^3)}{32\omega_1^2}(o_1 + 2\omega_2)
\]

\[
\times p_2(p_2^3 + p_4^3)\right\},
\]

\[
\dot{\hat{p}}_2 = -\frac{1}{2} c_1 p_2 + (2\sigma_2 + \sigma_1) p_1 - \frac{3\omega_2^2}{8\omega_1} (p_3^2 - p_4^2)
\]

\[
+ \left\{ -\frac{(4\omega_1 - 3\omega_2) f_2}{16\omega_1^2} p_3 + \frac{3\sigma_1 \omega_2^2}{16\omega_1^2} (p_3^2 - p_4^2) + \frac{c_1^2}{8\omega_1} p_1
\]

\[
- \frac{\omega_2(8c_2 \omega_1 + 3c_1 \omega_2 - 6c_2 \omega_2)}{16\omega_1^2} p_3 p_4
\]

\[
- \frac{\omega_2(8\omega_1^3 + 14\omega_1^2 \omega_2 - 17\omega_1 \omega_2^2 + 10\omega_2^3)}{32\omega_1^2}(o_1 + 2\omega_2)
\]

\[
\times (p_3^2 + p_4^3)\right\},
\]
Substituting Eqs. (11), (13), (17), (18) into Eq. (5),

\[
\dot{p}_3 = -\frac{1}{2} c_2 p_3 - \sigma_2 p_4 - \frac{1}{4}(2\omega_1 - \omega_2)(p_1 p_4 - p_2 p_3) \\
+ \left\{ -\frac{c_2^2}{8\omega_2} p_4 + \frac{c_2 f_2}{8\omega_2} + \frac{\sigma_1 (2\omega_1 - \omega_2)}{8\omega_2} \times (p_1 p_4 - p_2 p_3) + \frac{3(3\omega_2 - 2\omega_1)f_2}{16\omega_2^2} p_2 \\
+ \frac{(2c_1 \omega_1 + 4c_2 \omega_1 - 5c_1 \omega_2 - 8c_2 \omega_2)}{16\omega_2} \times \frac{p_3^2 + p_2^2}{p_1 + p_2} \right\},
\]

\[
\dot{p}_4 = -\frac{1}{2} c_2 p_4 + \sigma_2 p_3 - \frac{1}{4}(2\omega_1 - \omega_2)(p_1 p_3 + p_2 p_4) \\
+ \frac{f_2}{2\omega_2} + \left\{ -\frac{\sigma_2 f_2}{4\omega_2^2} + \frac{c_2^2}{8\omega_2} p_3 + \frac{3(2\omega_1 - 3\omega_2)f_2}{16\omega_2^2} p_1 \\
+ \frac{\sigma_1 (2\omega_1 - \omega_2)}{8\omega_2} (p_1 p_3 + p_2 p_4) + \frac{(2c_1 \omega_1 + 4c_2 \omega_1 - 5c_1 \omega_2 - 8c_2 \omega_2)}{16\omega_2} \times \frac{p_3^2 + p_2^2}{p_1 + p_2} \right\}. \tag{19}
\]

Substituting Eqs. (11), (13), (17), (18) into Eq. (5), and neglecting terms of \(O(\epsilon^3)\), we have

\[
x = p_1 \cos 2\Omega t + p_2 \sin 2\Omega t + \left[ \frac{f_1}{\omega_1^2 - \Omega^2} \cos \Omega t \right. \\
+ \frac{\omega_2^2}{4\omega_1^2} (p_3^2 + p_4^2) \left. \right] \\
= a_1 \cos (2\Omega t - \gamma_2 - 2\gamma_1) \\
+ \left[ \frac{f_1}{\omega_1^2 - \Omega^2} \cos \Omega t + \frac{\omega_2^2}{4\omega_1^2} \cos \Omega t \right] \\
\times \left\{ (p_1 p_3 - p_2 p_4) \cos 3\Omega t \\
+ (p_1 p_4 + p_2 p_3) \sin 3\Omega t \right\} \left[ \frac{1}{\omega_1^2 - \Omega^2} \cos \Omega t \right].
\]

Terms in brackets \{ \} of Eq. (19) and in brackets \[ \] of Eq. (20) are added because of the second-order approximation.

3. Numerical results

To obtain the numerical results, we have chosen values for the system parameters as follows: \(\omega_2 = 0.5, \ c_1 = 0.005, \ f_1 = 0.0\). In view of Eq. (20), we can see that each equilibrium solution of the approximate system (19) corresponds to each periodic solution of the original system (1).

The modal amplitudes \(a_1\) and \(a_2\) of the periodic solutions as functions of the forcing amplitude \(f_2\) are shown in Fig. 2, where solid and dotted lines denote stable and unstable solutions, respectively. Fig. 2a and Fig. 2b represent the first- and second-order approximation results, respectively. There exists one steady-state periodic solution corresponding to a coupled-mode solution, which loses its stability at the Hopf bifurcation point \(f_2 = f_{2H}\). We can easily see that there exists the non-linear modal interaction between the spring and pendulum modes by observing that the spring motion does not vanish, although it is not a directly excited mode. Comparing Fig. 2a and Fig. 2b we find that the contribution of the second-order approximation is not apparent as far as concerned with periodic solutions.

According to Lee and Park [15], when \(f_2 > f_{2B}\), there is no stable steady-state periodic solution,
thus the long-term response is aperiodic. That is, the motion becomes either quasi-periodic or chaotic. While the equilibrium solution of the approximate system (19), which corresponds to the coupled-mode periodic solution, loses its stability at \( f_2 = f_{2B} \), a stable periodic solution of system (19) corresponding to a quasi-periodic solution of the original system (1) is generated. In order to examine the behavior of the periodic solution of the system (19) for \( f_2 > f_{2B} \), we generate a Poincaré map in the space (p-phase space) by using a hyperplane \( p_2 = -0.2p_1 + 0.037 \). The bifurcation diagram in Fig. 3 shows the relation between \( f_2 \) and \( p_1 \) of periodic solution in the map. The diagram also shows the period-doubling bifurcation leading to the chaotic motions. Fig. 3a and Fig. 3b represent the first- and second-order approximation results, respectively. The differences between both results are very obvious in view of the regions where the periodic solutions exist. Especially the existence of the period-3 motion in Fig. 3b, which leads another sequence of period-doubling bifurcations, is notable. In order to characterize quantitatively the aperiodic motions, we calculate the largest Lyapunov exponent of long-term responses for the approximate and original systems. The aperiodic response is known to be chaotic when the corresponding exponent is positive. The largest Lyapunov exponents of long-term responses versus \( f_2 \) are shown in Fig. 4a – Fig. 4c which, respectively, correspond to the first-order approximate system [15], the second-order approximate system (19) and the original system (1).

In order to examine the effects of other important parameters, we plotted the largest Lyapunov exponents versus the internal detuning \( \sigma_1 \), the external detuning \( \sigma_2 \), and damping coefficient of the directly excited mode \( c_2 \), in Figs. 5–7, respectively. From any of the Figs. 4–7, we can say that the second-order approximation gives better qualitative agreement with the original system than the first-order approximation does.
Fig. 4. The largest Lyapunov exponents of the long-term responses versus the amplitude of the excitation $f_2$ for $(\omega_2, \sigma_1, \sigma_2, c_1, c_2, f_1) = \{0.5, 0.01, 0.005, 0.005, 0.005, 0.0\}$. (a) The first-order approximation; (b) the second-order approximation; (c) the original system.

Fig. 5. The largest Lyapunov exponents of the long-term responses versus the internal detuning $\sigma_1$, for $(\omega_2, f_2, \sigma_2, c_1, c_2, f_1) = \{0.5, 0.00086, 0.005, 0.005, 0.005, 0.0\}$. (a) The first-order approximation; (b) the second-order approximation; (c) the original system.

4. Concluding remarks

In order to investigate the influence of a higher-order approximation on the chaotic responses for a weakly non-linear multi-degree-of-freedom system, we examined a harmonically excited spring–pendulum system with internal resonance. The original system of non-autonomous differential equations governing the spring–pendulum system is reduced to an approximate system of autonomous differential equations by the method of multiple scales to the second order. By examining the bifurcation diagrams and the largest Lyapunov exponents, we observed that the second-order approximation gives better qualitative agreement with the original system than the first-order approximation does.
Fig. 6. The largest Lyapunov exponents of the long-term responses versus the external detuning $\sigma_2$, for $\{\omega_2, f_2, \sigma_1, c_1, c_2, f_1\} = [0.5, 0.00086, 0.01, 0.005, 0.005, 0.0]$. (a) The first-order approximation; (b) the second-order approximation; (c) the original system.

Fig. 7. The largest Lyapunov exponents of the long-term responses versus the damping coefficient of the pendulum mode $c_2$, for $\{\omega_2, f_2, \sigma_1, \sigma_2, c_1, f_1\} = [0.5, 0.00089, 0.01, 0.005, 0.005, 0.0]$. (a) The first-order approximation; (b) the second-order approximation; (c) the original system.

References


