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Fault Diameter of Even Networks

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Abstract

A. Ghafoor proposed Even networks as a class of fault-tolerant multiprocessor networks in [8] and analyzed so many useful properties include node-disjoint paths. By introducing node-disjoint paths, fault diameter of Even networks can be $d+2$ (d =odd) or $d+3$ (d =even). However, the lengths of node-disjoint paths are not the shortest. In this paper, we show that Even networks are node- and edge- symmetric. We also propose the shortest lengths of node-disjoint paths, and show that fault diameter of Even networks is $d+1$.

1. Introduction

One of the central problems in computer networks research is to design network topologies that have good properties. These properties can be grouped into two major categories: better performance and lower cost. Cost usually refers to the diameter and the number of the links, and performance usually refers to fault tolerance, broadcast time, or ease of routing schemes. The best performance can be achieved when all nodes of a graph are connected to all the rest, i.e in a complete graph. But this is practically impossible and very costly. A compromise between cost and performance should be done and that is why there is research to desing networks with good structural characteristics as well as near optimum performance.

Symmetry is an important feature for most graph models used for interconnection networks. In a symmetric interconnection network, the load can be evenly distributed through all nodes, reducing congestion problems. Moreover, symmetry makes the design of routing algorithms easier because it allows routing between any two nodes to be mapped to routing between an arbitrary node and a specific node.

A common notion of fault tolerance in interconnection networks is based on the connectivity of the underlying graph. In an interconnection network with node connectivity of n, the graph is guaranteed to remain connected even if $n-1$ node processors fail. However, while the connectivity of such a network is still preserved, the network diameter may increase significantly. A good measure to judge this fault tolerance aspect of the network is the fault diameter. The fault diameter of many well-known networks have been determined by several researchers, see, for example, [1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 15, 16]. The concept of fault diameter was first proposed by Krishnamoorthy and Krishnamurthy [10]. The fault diameter of an interconnection network G is the maximum length of the shortest paths between all two fault-free nodes when there are $k(G)-1$ or less faulty nodes, where $k(G)$ is the connectivity of G. The fault diameter of an r-regular graph with diameter of 3 or more and connectivity r is at least $\mathcal{D}$+1 where $\mathcal{D}$ is the diameter of G.

In [8], Ghafoor introduced even network E_d by a class of fault-tolerant multiprocessor networks. In his study some important properties such as maximal fault-tolerance, node disjoint paths, routing algorithms during fault-free and faulty conditions, and ease of self-diagnosis were analyzed [8, 9]. He showed node-disjoint paths as follows:

Theorem 1 *The number of node-disjoint paths between any two nodes $x, y \in E_d$ is the maximum possible and is equal to d. The lengths of such paths are:*
Case a) L_{xy} is even: There are $\frac{L_{xy}}{2}$ paths of length L_{xy}. The remaining paths are of equal length, which is $L_{xy}+2$.
Case b) L_{xy} is odd: There are $\frac{(L_{xy}+1)}{2}$ paths of length L_{xy}. There is one alternate path of length L_{xy}+2. The remaining paths are of length $L_{xy}+4$.

By theorem 1, the lengths of node-disjoint paths are not the shortest and the fault diameter of E_d is d+2 (d =odd) or

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$d+3$ (d =even). For example, let L_{xy} be 3 in E_4. There are 2 paths of length 3, 1 path of length 5 and 1 path of length 7, and fault diameter of E_4 is 7 ($= d+3$) by theorem 1. However, we can find 4 paths of length 3 in the above case and reduce fault diameter of E_4. An example will be shown in section 3. In this paper, we show that Even networks are node- and edge- symmetric. We also propose the shortest lengths of node-disjoint paths and show that fault diameter is $d+1$.

2. Preliminaries

In [8], Ghafoor introduced even network E_d by a class of fault-tolerant multiprocessor networks. Even networks have some good properties such as maximal fault-tolerance, simple routing alforithms, combinatorial structures using hadamard matrix and semi-distributed fault-tolerant diagnostic algorithm. Even network E_d with $d \geq 2$ has the set of binary strings of length $2d-3$ with $|1| = |0| \pm 1$ as the node set. The number of nodes in E_d is $\binom{2d-2}{d-1}$, degree of E_d is d and its diameter $\mathcal{D}$ is $d-1$. Two nodes are adjacent if and only if their Hamming distance is 1 or $2d-3$. The *Hamming distance*, denoted as H_{xy}, between two binary strings, x and y, is the number of positions at which these strings differ. A *path* in E_d is a sequence of connected nodes. The *graphical distance*, denoted as L_{xy} (=min$(H_{xy}, 2d-2-H_{xy})$), is the length of the shortest path between two nodes x and y. An edge connecting two nodes u and v is denoted as i-edge and c-edge, where Hamming distance is 1 and $2d-3$. Nodes of the Even network represented by a bit string $s_1s_2...s_{2d-3}$ can be divided into two sets, S_d^1 and S_d^0. S_d^1 is the set of nodes such that $|1| = |0| + 1$ and S_d^0 is the set of nodes such that $|0| = |1| + 1$. An arbitrary node $u \in S_d^1$ is only connected a node $v \in S_d^0$. Because the edge connects two nodes in the case where H_{uv} is 1 or $2d-3$. Thus, E_d is a bipartite graph and has only an even length of cycles if it contains cycles. We write a node $\overbrace{0\ldots0}^{d-2}\overbrace{1\ldots1}^{d-1}$ in E_d as $0^{d-2}1^{d-1}$. Fig. 1 shows an Even network E_4.

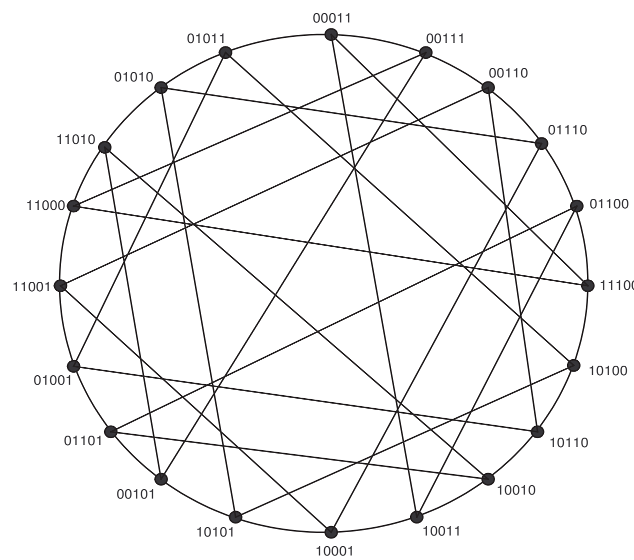

Figure 1. E_4

3. Fault Diameter of Even Networks

A graph G is said to be *node-symmetric* if, for any two nodes u and v, there exists an automorphism of the graph G that maps u into v. In other words, G has the same shape when viewed from any node.

Theorem 2 *Even network E_d is node- and edge- symmetric.*

Proof Let $u = s_1s_2...s_i...s_{2d-3}$, $v = s_1s_2...\bar{s}_i...s_{2d-3}$, $v' = \bar{s}_1\bar{s}_2...\bar{s}_i...\bar{s}_{2d-3}$, $w = s_1s_2...\bar{s}_i...\bar{s}_j...s_{2d-3}$, $w' = \bar{s}_1\bar{s}_2...s_i...\bar{s}_j...\bar{s}_{2d-3}$.
Case 1) E_d is node-symmetric; We may assume that two adjacent nodes are u and v (or v'). Let ϕ be the function that switches all (or some) 0s to 1s and vice versa for every node in E_d. Then, $\phi(u) = \bar{s}_1\bar{s}_2...\bar{s}_i...\bar{s}_{2d-3}$ and $\phi(v) = \bar{s}_1\bar{s}_2...s_i...\bar{s}_{2d-3}$ (or $\phi(v') = s_1s_2...s_i...s_{2d-3}$). It is easy to check that $\phi(u)$ and $\phi(v)$ (or $\phi(v')$) are adjacent. Hence E_d is node-symmetric.
Case 2) E_d is edge-symmetric; We may assume that the two adjacent edges are $e_1 = (u, v \text{ (or } v'))$ and $e_2 = (v, w)$ (or (v', w')). Let ψ be the bijective function defined by $\psi(u) = w$ (or w') for every node in E_d. Then it is easy to check that ψ preserves the adjacency and maps the end-node of e_1 to the end-node of e_2. Hence E_d is edge-symmetric.

Fault tolerance of a network is defined in terms of the smallest width container in the network, where a container is defined to be the set of node-disjoint paths between a pair of nodes [14]. In order to find fault tolerance for Even networks and the width of their containers, let us define P_{ij}^{uv} to be the set of positions in the bitstrings associated with nodes u and v, such that if u has bit value i, then v has bit value j ($i, j = 0, 1$). Also, let T_k be an operator which, when it operates on a bitstring u, yields the bitstring of a neighboring node v, with which u has the kth bit complemented. Furthermore, if the operator T_t has $t \in P_{ij}^{xy}$, we will denote it as T_t^{ij}. Equivalently we can also represent the edge between two bitstrings having the kth bit complemented, as T_k. Therefore, a path between any two nodes in the network is representable as a sequence of T_k's.

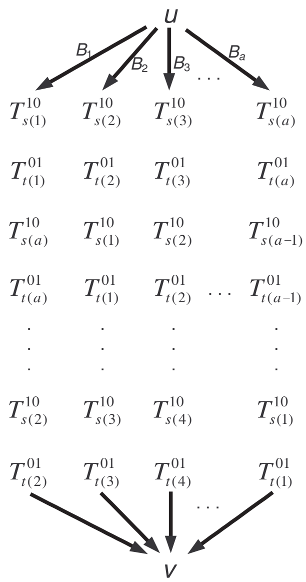

Figure 2. $T^{10}_{s(i)} \odot T^{01}_{t(j)}$

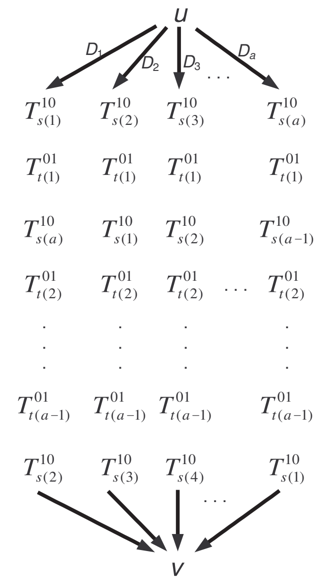

Figure 3. $T^{10}_{s(i)} \odot T^{01-}_{t(j)}$

Consider the cyclic permutation of two sequences $S_1 = (a_1, a_2, \ldots, a_p)$ and $S_2 = (b_1, b_2, \ldots, b_q)$ denoted by $S_1 \odot S_2$. $S_1 \odot S_2$ is the set of sequences obtained by merging symbols in S_1 and S_2 alternately. If only one sequence is permuted, say S_2, then we write $S_1 \odot S_2^-$. Also, If c is included in $S_1 \odot S_2$ or $S_1 \odot S_2^-$, then it is denoted by $c \diamond S_1 \odot S_2$ and $c \diamond S_1 \odot S_2^-$. Let two nodes be $u = 0^{d-2}1^{d-1}$ and v in E_d. Then fig. 2, 3, 4 and 5 show $T^{10}_{s(i)} \odot T^{01}_{t(j)}$, $T^{10}_{s(i)} \odot T^{01-}_{t(j)}$, $c \diamond T^{11}_{s(i)} \odot T^{00}_{t(j)}$, $c \diamond T^{11}_{s(i)} \odot T^{00-}_{t(j)}$ $(1 \leq i,\ j \leq a)$, respectively. If L_{uv} is even, then $a = \frac{L_{uv}}{2}$, else $a = \frac{L_{uv}+1}{2}$. We denote each path in $T^{10}_{s(i)} \odot T^{01}_{t(j)}$, $T^{10}_{s(i)} \odot T^{01-}_{t(j)}$ as B_x, D_x $(1 \leq x \leq a)$ and each path in $c \diamond T^{11}_{s(i)} \odot T^{00}_{t(j)}$, $c \diamond T^{11}_{s(i)} \odot T^{00-}_{t(j)}$ as G_x, F_x $(0 \leq x \leq a)$. If L_{uv} is 1, then G_0 and F_0 are the same as G_1 and F_1, respectively, as c.

lemma 1 *An arbitrary sequence proposed of T_i's and T_c in E_d constitutes a cycle $(1 \leq i \leq 2d-3)$.*

Proof Let u be an arbitrary node in E_d. An arbitrary node u is connected to its complementary node $\bar{u}$ by T_i's $(1 \leq i \leq 2d-3)$. And $\bar{u}$ is connected to u by T_c. Hence, the proof is completed.

Theorem 3 *All of the paths B_x $(1 \leq x \leq a)$ in $T^{10}_{s(i)} \odot T^{01}_{t(j)}$ are node-disjoint.*

Proof Since E_d is node-symmetric, let two given nodes be $u = 0^{d-2}1^{d-1}$ and v. As shown in figure 2, these paths

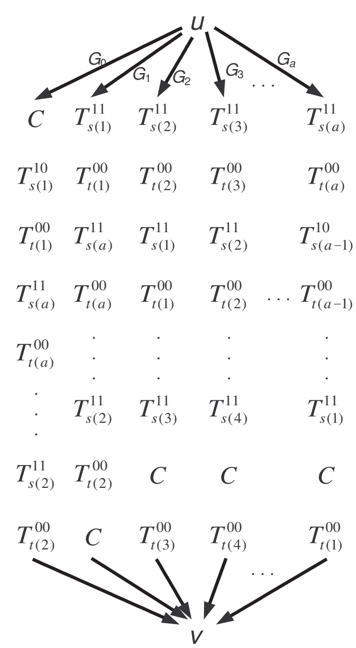

Figure 4. $c \diamond T^{11}_{s(i)} \odot T^{00}_{t(j)}$

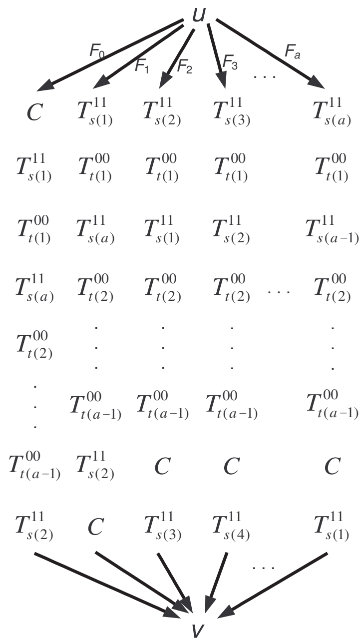

Figure 5. $c \diamond T^{11}_{s(i)} \odot T^{00-}_{t(j)}$

are permuted sequences of the operators T_i's from u to v. In these paths, operators of the same type, say $T^{10}_{s(i)}$, appear at the odd levels, while the others appear at the even levels. These paths are of the shortest possible length, because the selection of the operators T_i's in each path is consistent with the shortest path routing algorithm. Consider two paths in fig. 2, say B_1 and B_i, where B_i is some ith cyclically permuted version of B_1. Suppose there is a common node w ($\neq u, v$) in two paths. Then, the selection of the operators T_i's from u to w in two paths must be the same. However, this is impossible, because B_i is some ith cyclically permuted version of B_1. Therefore, there is no common node w ($\neq u, v$) in two paths. So, both B_1 and B_i are node-disjoint. Similarly, it can be proven that all of the paths in fig. 3, 4 and 5 are node-disjoint.

By lemma 1 and theorem 3, All of the paths in $T^{10}_{s(i)} \odot T^{01}_{t(j)}$ and in $c \diamond T^{11}_{s(i)} \odot T^{00-}$ are node-disjoint, and all of the paths in $T^{10}_{s(i)} \odot T^{01-}_{t(j)}$ and in $c \diamond T^{11}_{s(i)} \odot T^{00}_{t(j)}$ are node-disjoint too.

We can easily check that fault diameter of E_3 is 3. Therefore, we will prove fault diameter of E_d, $d \geq 4$.

Theorem 4 *Fault diameter of* $E_d = d + 1$ $(d \geq 4)$.

Proof Since E_d is node-symmetric, let two given nodes be $u = 0^{d-2}1^{d-1}$ and $v = v_1 v_2 \ldots v_i \ldots v_{2d-3}$ in E_d.

Case 1)L_{uv} =odd; For x in this case, $1 \leq x \leq \frac{L_{uv}+1}{2}$.

Case 1-1)$L_{uv} = H_{uv} = \mathcal{D}$: There are $\frac{L_{uv}+1}{2}$ paths of the form D_x and $d - \frac{L_{uv}+1}{2}$ paths of the form G_x of length L_{uv}. So, the length of the longest path is $L_{uv} = d - 1$.

Case 1-2)$L_{uv} = 2d-2-H_{uv} = \mathcal{D}-1$: There are $\frac{L_{uv}+1}{2}$ paths of the form G_x and $d - \frac{L_{uv}+1}{2}$ paths of the form D_x of length $L_{uv} + 2$. So, the length of the longest path is $L_{uv} + 2 \leq d$.

Case 1-3)$L_{uv} = H_{uv} = \mathcal{D}-1$: There are $\frac{L_{uv}+1}{2}$ paths of the form D_x and $d - \frac{L_{uv}+1}{2}$ paths of the form G_x of length $L_{uv}+2$. So, the length of the longest path is $L_{uv}+2 \leq d+1$.

Case 1-4)$L_{uv} = 2d-2-H_{uv}$: There are $\frac{L_{uv}+1}{2}$ paths of the form G_x and $d - \frac{L_{uv}+1}{2}$ paths of the form $T^{10}_{s(h)}, P, T^{10}_{s(h)}$ of length $L_{uv} + 2$ $(1 \leq h \leq d - \frac{L_{uv}+1}{2})$. P is the path of G_0 in reverse order. So, the length of the longest path is $L_{uv} + 2 \leq d - 1$.

Case 1-5)$L_{uv} = H_{uv}$: There are $\frac{L_{uv}+1}{2}$ paths of the form D_x and 1 path of the form c, D_1, c of length $L_{uv} + 2$ and $d - 1 - \frac{L_{uv}+1}{2}$ paths of the form $T^{11}_{s(h)}, T^{00}_{t(h)}, D_1, T^{11}_{s(h)}, T^{00}_{t(h)}$ of length $L_{uv} + 4$ $(1 \leq h \leq d - 1 - \frac{L_{uv}+1}{2})$. So, the length of the longest path is $L_{uv} + 4 \leq d + 1$.

Case 2)L_{uv} =even; For x in this case, $1 \leq x \leq \frac{L_{uv}}{2}$.

Case 2-1)$L_{uv} = H_{uv} = \mathcal{D}$: There are $\frac{L_{uv}}{2}$ paths of the form B_x and $d - \frac{L_{uv}+1}{2}$ paths of the form F_x of length L_{uv}. So, the length of the longest path is $L_{uv} = d - 1$.

Case 2-2)$L_{uv} = 2d - 2 - H_{uv} = \mathcal{D}-1$: There are $\frac{L_{uv}}{2}$ paths of the form F_x and $d - \frac{L_{uv}}{2}$ paths of the form B_x of length $L_{uv} + 2$. So, the length of the longest path is $L_{uv} + 2 \leq d$.

Case 2-3)$L_{uv} = 2d - 2 - H_{uv}$: There are $\frac{L_{uv}}{2}$ paths of the form F_x and $d - \frac{L_{uv}}{2}$ paths of the form $T^{10}_{s(h)}, T^{01}_{t(h)}, F_1, T^{10}_{s(h)}, T^{01}_{t(h)}$ of length $L_{uv} + 4$ $(1 \leq h \leq d - \frac{L_{uv}}{2})$. So, the length of the longest path is $L_{uv} + 4 \leq d + 1$.

Case 2-4)$L_{uv} = H_{uv}$: There are $\frac{L_{uv}}{2}$ paths of the form B_x and 1 path of the form c, B_1, c of lenth $L_{uv} + 2$ and $d - 1 - \frac{L_{uv}}{2}$ paths of the form $T^{11}_{s(h)}, P, T^{11}_{s(h)}$ of length $L_{uv} + 2$ $(1 \leq h \leq d - 1 - \frac{L_{uv}}{2})$. P is the path of B_1 in reverse order. So, the length of the longest path is $L_{uv} + 2 \leq d - 1$.

All of the paths are node-disjoint, because all of the symbols $T^{10}_{s(h)}, T^{01}_{t(h)}, T^{11}_{s(h)}, T^{00}_{t(h)}$ are unique and lemma 1 and theorem 3. Hence Fault diameter of $E_d = d + 1$ $(d \geq 4)$. Also, these paths are the shortest paths. Because the shortest path distance is L_{uv}, the alternate paths must be of length greater than L_{uv}. E_d is a bipartite graph and it cannot contain an odd cycle. So, the next path distance is

$L_{uv}+2$ or $L_{uv}+4$. In the case of 1-5, 2-3, the reason that the length of the alternate paths is $L_{uv}+4$ is first operator in D_1, F_1 is $T^{10}_{s(1)}, T^{11}_{s(1)}$.

For the comparison of theorem 1 and theorem 4, we show two examples. One is $L_{uv} = 3$ in E_5, and the other is $L_{uv} = 5$ in E_6.

Example 1) Let $u = 0001111$, $v = 1001001$ in E_5, then the Hamming distance H_{uv} is 3, and $L_{uv} = H_{uv} = 3$.

Case 1) according to theorem 1 : there must exist 5 node-disjoint paths between nodes u and v, 2 paths of length of 3, 1 path of length of 5 and 2 paths of length of 7. In order to find these paths the sets $P^{u\bar{v}}_{11} = \{5,6\}$, $P^{u\bar{v}}_{00} = \{1\}$, $P^{u\bar{v}}_{10} = \{4,7\}$ and $P^{u\bar{v}}_{01} = \{2,3\}$ are needed. Using these sets and theorem 4, the sequences of operators and the paths are as follows:

$$\text{length of 3} : 0001111 \overset{T_5}{-} 0001011 \overset{T_1}{-} 1001011 \overset{T_6}{-} 1001001$$
$$\text{length of 3} : 0001111 \overset{T_6}{-} 0001101 \overset{T_1}{-} 1001101 \overset{T_5}{-} 1001001$$
$$\text{length of 5} : 0001111 \overset{c}{-} 1110000 \overset{T_5}{-} 1110100 \overset{T_1}{-} 0110100 \overset{T_6}{-} 0110110 \overset{c}{-} 1001001$$
$$\text{length of 7} : 0001111 \overset{T_4}{-} 0000111 \overset{T_2}{-} 0100111 \overset{T_5}{-} 0100011 \overset{T_1}{-} 1100011 \overset{T_6}{-} 1100001 \overset{T_4}{-} 1101001 \overset{T_2}{-} 1001001$$
$$\text{length of 7} : 0001111 \overset{T_7}{-} 0001110 \overset{T_3}{-} 0011110 \overset{T_5}{-} 0011010 \overset{T_1}{-} 1011010 \overset{T_6}{-} 1011000 \overset{T_7}{-} 1011001 \overset{T_3}{-} 1001001$$

Case 2) according to case 1-3 in theorem 4 : there must exist 5 node-disjoint paths between nodes u and v, 2 paths of length of 3 and 3 paths of length of 5. In order to find these paths the sets $P^{uv}_{11} = \{4,7\}$, $P^{uv}_{00} = \{2,3\}$, $P^{uv}_{10} = \{5,6\}$ and $P^{uv}_{01} = \{1\}$ are needed. Using these sets and theorem 4, the sequences of operators and the paths are as follows:

$$D_1 : 0001111 \overset{T_5}{-} 0001011 \overset{T_1}{-} 1001011 \overset{T_6}{-} 1001001$$
$$D_2 : 0001111 \overset{T_6}{-} 0001101 \overset{T_1}{-} 1001101 \overset{T_5}{-} 1001001$$
$$G_1 : 0001111 \overset{c}{-} 1110000 \overset{T_4}{-} 1111000 \overset{T_2}{-} 1011000 \overset{T_7}{-} 1011001 \overset{T_3}{-} 1001001$$
$$G_2 : 0001111 \overset{T_4}{-} 0000111 \overset{T_2}{-} 0100111 \overset{T_7}{-} 0100110 \overset{T_3}{-} 0110110 \overset{c}{-} 1001001$$
$$G_3 : 0001111 \overset{T_7}{-} 0001110 \overset{T_3}{-} 0011110 \overset{T_4}{-} 0010110 \overset{c}{-} 1101001 \overset{T_2}{-} 1001001$$

Example 2) Let $u = 000011111$, $v = 100101010$ in E_6, then the Hamming distance H_{uv} is 5, and $L_{uv} = H_{uv} = 2d-2-H_{xy} = 5$.

Case 1) according to theorem 1 : there must exist 6 node-disjoint paths between nodes u and v, 3 paths of length of 5, 1 path of length of 7 and 2 paths of length of 9. In order to find these paths the sets $P^{u\bar{v}}_{11} = \{5,7,9\}$, $P^{u\bar{v}}_{00} = \{1,4\}$, $P^{u\bar{v}}_{10} = \{6,8\}$ and $P^{u\bar{v}}_{01} = \{2,3\}$ are needed. Using these sets and theorem 4, the sequences of operators and the paths are as follows:

$$\text{length of 5} : 000011111 \overset{T_5}{-} 000001111 \overset{T_1}{-} 100001111 \overset{T_9}{-} 100001110 \overset{T_4}{-} 100101110 \overset{T_7}{-} 100101010$$
$$\text{length of 5} : 000011111 \overset{T_7}{-} 000011011 \overset{T_4}{-} 000111011 \overset{T_5}{-} 000101011 \overset{T_1}{-} 100101011 \overset{T_9}{-} 100101010$$
$$\text{length of 5} : 000011111 \overset{T_9}{-} 000011110 \overset{T_1}{-} 100011110 \overset{T_7}{-} 100011010 \overset{T_4}{-} 100111010 \overset{T_5}{-} 100101010$$
$$\text{length of 7} : 000011111 \overset{c}{-} 111100000 \overset{T_5}{-} 111110000 \overset{T_1}{-} 011110000 \overset{T_9}{-} 011110001 \overset{T_4}{-} 011010001 \overset{T_7}{-} 011010101 \overset{c}{-} 100101010$$
$$\text{length of 9} : 000011111 \overset{T_6}{-} 000010111 \overset{T_2}{-} 010010111 \overset{T_5}{-} 010000111 \overset{T_1}{-} 110000111 \overset{T_9}{-} 110000110 \overset{T_4}{-} 110100110 \overset{T_7}{-} 1101100010 \overset{T_6}{-} 110101010 \overset{T_2}{-} 100101010$$
$$\text{length of 9} : 000011111 \overset{T_8}{-} 000011101 \overset{T_3}{-} 001011101 \overset{T_5}{-} 001001101 \overset{T_1}{-} 101001101 \overset{T_9}{-} 101001100 \overset{T_4}{-} 101101100 \overset{T_7}{-} 101101000 \overset{T_8}{-} 101101010 \overset{T_3}{-} 100101010$$

Case 2) according to case 1-1 in theorem 4 : there must exist 6 node-disjoint paths between nodes u and v, 6 paths of length of 5. In order to find these paths the sets $P^{uv}_{11} = \{6,8\}$, $P^{uv}_{00} = \{2,3\}$, $P^{uv}_{10} = \{5,7,9\}$ and $P^{uv}_{01} = \{1,4\}$ are needed. Using these sets and theorem 4, the sequences of operators and the paths are as follows:

$$D_1 : 000011111 \overset{T_5}{-} 000001111 \overset{T_1}{-} 100001111 \overset{T_7}{-} 100001011 \overset{T_4}{-} 100101011 \overset{T_9}{-} 100101010$$
$$D_2 : 000011111 \overset{T_7}{-} 000011011 \overset{T_1}{-} 100011011 \overset{T_9}{-} 100011010 \overset{T_4}{-} 100111010 \overset{T_5}{-} 100101010$$
$$D_3 : 000011111 \overset{T_9}{-} 000011110 \overset{T_1}{-} 100011110 \overset{T_5}{-} 100001110 \overset{T_4}{-} 100101110 \overset{T_7}{-} 100101010$$
$$G_1 : 000011111 \overset{c}{-} 111100000 \overset{T_6}{-} 111101000 \overset{T_2}{-} 101101000 \overset{T_8}{-} 101101010 \overset{T_3}{-} 100101010$$
$$G_2 : 000011111 \overset{T_6}{-} 000010111 \overset{T_2}{-} 010010111 \overset{T_8}{-} 010010101 \overset{T_3}{-} 011010101 \overset{c}{-} 100101010$$
$$G_3 : 000011111 \overset{T_8}{-} 000011101 \overset{T_3}{-} 001011101 \overset{T_6}{-} 001010101 \overset{c}{-} 110101010 \overset{T_2}{-} 100101010$$

By theorem 1, the sets of node-disjoint paths in the above examples are the paths that have the longest length in E_5 and E_6. This means that the fault diameters of each E_5 and E_6 are each 7 and 9 according to theorem 1. However, those node-disjoint paths obtained by theorem 4 in the examples are not the paths that have the longest length of E_5 and E_6. This implies that the longest node-disjoint paths in E_5 and E_6 are the paths obtained by the case 2-3 and the case 1-5 in theorem 4; therefore each fault diameter of E_5 and E_6 is each 6 and 7. In conclusion, it is clear that the fault diameter of E_d obtained by theorem 4 is better than that obtained by theorem 1.

4. Conclusion

In this paper, we showed that E_d is node- and edge- symmetric. Further studies on the other properties of E_d such as its Hamiltonicity would be interesting. We also proved the shortest lengths of node-disjoint paths of E_d and showed that fault diameter of E_d is $d + 1$. It has been shown that node-disjoint paths and fault diameter derived in this paper are better than the previously known bound.

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