On Gabor frames generated by sign-changing windows and B-splines

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Abstract

For a class of compactly supported windows we characterize the frame property for a Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, for translation parameters a belonging to a certain range depending on the support size. We show that the obstructions to the frame property are located on a countable number of "curves." For functions that are positive on the interior of the support these obstructions do not appear, and the considered region in the (a, b) plane is fully contained in the frame set. In particular this confirms a recent conjecture about B-splines by Gröchenig in that particular region. We prove that the full conjecture is true if it can be proved in a certain "hyperbolic strip."

Keywords: Gabor frames; frame set; B-splines

1 Introduction

Only for quite special functions $g \in L^2(\mathbb{R})$ we know a characterization of the *Gabor frame set,* $\mathcal{F}(g) := \{(a, b) \in \mathbb{R}_+^2 \,\big|\, \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \text{ is a frame}\}$; these functions include the Gaussian [14, 18], the hyperbolic secant [10], the one-sided/two-sided exponentials [11], and totally positive functions [8]. Common for all these functions is that they are nonnegative.

Much less is known about more general functions, e.g., functions that change sign. In this paper we consider a class of continuous compactly supported windows g with $\operatorname{supp} g = [-\alpha, \alpha]$ for some $\alpha > 0$ and characterize the

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frame property of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in the region $\alpha \le a < 2\alpha, b < 1/a$. For technical reasons (and in order to avoid pathological examples of no practical interest) we assume that the function g only has a finite number of zeros in $]-\alpha,\alpha[$. The general result, to be stated in Theorem 2.1, shows that the zeros in the interior of the support lead to certain obstacles for the frame property that cannot be predicted from the known results for nonnegative functions. For each translation parameter a countable number of obstructions can appear, i.e., one can think about the obstructions as located on a countable number of curves in the (a,b)–plane. The general result also implies the existence of a compactly supported dual window if the frame property is satisfied, with an interesting interpretation in terms of the redundancy of the frame: in fact, if $\frac{M-1}{M} \le ab < \frac{M}{M+1}$ for some $M = 2, 3, \dots,$ i.e., if the redundancy $(ab)^{-1}$ is at least $\frac{M+1}{M} = 1 + 1/M$, the existence of a dual window supported on $[-2\alpha M, 2\alpha M]$ is guaranteed.

In the special case of a function g that is positive on $]-\alpha,\alpha[$ the general result implies that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for all parameters a, b in the considered region $\alpha \le a < 2\alpha, b < 1/a$. In particular, any B-spline $B_N, N \ge 2$, generates a frame $\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ whenever $N/2 \le a < N$, $b < 1/a$. This confirms a recent conjecture by Gröchenig in that particular region. Inspired by this result we prove that the full conjecture holds if it can be verified in the region determined by the inequalities $1/2 \le ab < 1$, $a < N/2$.

The key result in the paper is Theorem 2.1, which characterizes the frame property of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in the aforementioned region. The proof is quite complicated and is split into several lemmas and intermediate steps. The idea of the proof was gained through the work on the special case with translation parameter $a = 1$ (see the paper [2]), as well as the observation that the duality condition (3.3) forces a certain behavior of the window g around points $x_0 + a$ for which $g(x_0) = 0$. As further help to understand the idea behind the proof we prove the steps directly in a concrete case, see Example 2.2. For more informations about Gabor systems and frames we refer to the monographs [5,1].

2 General results

Given $\alpha > 0$, let

$$V_\alpha := \{f \in C(\mathbb{R}) \mid \operatorname{supp} f = [-\alpha,\alpha],\ f \text{ has a finite number of zeros on } [-\alpha,\alpha]\}. \tag{2.1}$$

We first characterize the frame property of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for functions $g \in V_\alpha$ and points (a,b) in the region in $\mathbb{R}^2_+$ determined by the inequalities

$\alpha \leq a < 2\alpha, b < 1/a$. In order to do this, we need to introduce some parameters and other tools. Consider (a,b) belonging to the described region, and choose $M \in \mathbb{N}$ such that $ab \in [\frac{M-1}{M}, \frac{M}{M+1}[$. Let κ be the largest integer for which $(1-ab)\kappa \leq b\alpha$. Then $0 \leq \kappa \leq M-1$ because

$$\kappa \leq \frac{b\alpha}{1-ab} \leq \frac{ab}{1-ab} < \frac{M}{M+1}\left(1-\frac{M}{M+1}\right)^{-1} = M.$$

If $\kappa \neq 0$, let $n \in \{1,2,\cdots,\kappa\}$, and define the function R_n on (a subset of) $]a-\alpha, \alpha-(1-ab)\frac{n}{b}] \subset]a-\alpha,\alpha]$ by

$$R_n(y) := \frac{1}{g(y)} \prod_{k=1}^{n-1} \frac{g(y+(1-ab)\frac{k}{b}-a)}{g(y+(1-ab)\frac{k}{b})}, \quad n = 1,2,\cdots,\kappa. \tag{2.2}$$

We use the standard convention that the empty product is 1. It is easy to see that R_n indeed is defined on $]a-\alpha, \alpha-(1-ab)\frac{n}{b}]$, except possibly on a finite set of points. Similarly, still if $\kappa \neq 0$, for $n \in \{1,\cdots,\kappa\}$ we define the function $L_n(y)$ on (a subset of) $[-\alpha+(1-ab)\frac{n}{b}, \alpha-a[\subset [-\alpha, \alpha-a[$ by

$$L_n(y) := \frac{1}{g(y)} \prod_{k=1}^{n-1} \frac{g(y-(1-ab)\frac{k}{b}+a)}{g(y-(1-ab)\frac{k}{b})}, \quad n = 1,2,\cdots,\kappa.$$

We now state the announced characterization of the frame property.

Theorem 2.1 *Let $g \in V_\alpha$ for some $\alpha > 0$ and assume that $\alpha \leq a < 2\alpha$ and $ab \in [\frac{M-1}{M}, \frac{M}{M+1}[$ for some $M \in \mathbb{N} \setminus \{1\}$. Let $\kappa \in \{0,1,\cdots,M-1\}$ be the largest integer for which $(1-ab)\kappa \leq b\alpha$. Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame if and only if the following conditions are satisfied:*

(i) $|g(x)| + |g(x+a)| > 0, \ x \in [-a, 0]$;

(ii) *If $\kappa \neq 0$ and if there exist $n_+ \in \{1,2,\cdots,\kappa\}$ and $y_+ \in]a-\alpha, \alpha-(1-ab)\frac{n_+}{b}]$ such that $g(y_+) = 0$ and $\lim_{y\to y_+} |R_{n_+}(y)| = \infty$, then*

$$g(y_+ + (1-ab)\frac{n_+}{b} - a) \neq 0;$$

(iii) *If $\kappa \neq 0$ and if there exist $n_- \in \{1,2,\cdots,\kappa\}$ and $y_- \in [-\alpha+(1-ab)\frac{n_-}{b}, \alpha-a[$ such that $g(y_-) = 0$ and $\lim_{y\to y_-} |L_{n_-}(y)| = \infty$, then*

$$g(y_- - (1-ab)\frac{n_-}{b} + a) \neq 0;$$

(iv) *For y_+, y_-, n_+, n_- as in* (ii) *and* (iii)*,*

$$y_+ + (1-ab)\frac{n_+}{b} \neq y_- - (1-ab)\frac{n_-}{b} + a.$$

In the affirmative case, there exists a dual window h with $\mathrm{supp}\, h \subseteq [-aM, aM]$.

We remark that if $\kappa = 0$ then the conditions (ii)-(iv) are trivially satisfied. We also note that Theorem 2.1 is similar, but significantly more general than Theorem 2.3 in [2]. The main difference is that in the current paper the support size of g (measured by the parameter α) and the translation parameter a can vary, subject to the restriction $\alpha \le a < 2\alpha$; on the other hand [2] dealt with the case $\alpha = a = 1$. This modification turns out to be instrumental for our applications to B-splines.

The proof of the necessity of the conditions in Theorem 2.1 is similar to the proof in [2], so we skip this part. On the other hand, it requires much more work to prove that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame if the conditions (i)-(iv) are satisfied. We prove this part of the theorem in the appendix. In the subsequent example we prove directly that a certain Gabor system is a frame, following the steps from the proof of the general result; the hope is that the analysis of this concrete case will help the reader to understand the idea behind the general proof.

Example 2.2 Let $\alpha = 9/10$ and consider a function $g \in V_\alpha$, having the single zero $1/5$ within $]-1,1[$. Let $a = 1$ and $b = 3/5$. Then $ab \in [\frac{M-1}{M}, \frac{M}{M+1}[$ for $M = 2$. Note that $(1-ab)/b = 5/3 - 1 = 2/3 \le 9/10 = \alpha$. This implies that $\kappa = 1$. Trivially, $|g(x)| + |g(x+a)| > 0,\ x \in [-a, 0]$. Let $n_+ := 1$ and $y_+ := 1/5$. Then $y_+ \in]a-\alpha, \alpha - (1-ab)\frac{n_+}{b}] =]1/10, 7/30]$ and $g(y_+) = 0$. Furthermore, $R_{n_+}(y) = g(y)^{-1}$, so $\lim_{y\to y_+} |R_{n_+}(y)| = \infty$. It is also clear that $g(y_+ + (1-ab)\frac{n_+}{b} - a) = g(-2/15) \neq 0$.

It is an easy consequence of the duality conditions for Gabor frames (see (3.3) in the Appendix) that two real valued, bounded functions $g, h \in L^2(\mathbb{R})$ with $\mathrm{supp}\, h \subseteq [-aM, aM] = [-2, 2]$ generate dual frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ if and only if for $n = 0, \pm 1$ and *a.e.* $x \in [\frac{n}{b} - a, \frac{n}{b}]$,

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + a)h(x+a) = b\delta_{n,0}. \tag{2.3}$$

We will check (2.3) directly following the steps in the general proof of Theorem 2.1. Motivated by a general result, see Lemma 3.3, we choose to put $h(x) = 0$ for $x \notin [-a-\alpha, -\frac{1}{b}] \cup [-\alpha, \alpha] \cup [\frac{1}{b}, \alpha + a]$. Then $h(x) = h(x+a) = 0$ for $x \in]\alpha, \frac{1}{b}[$, which is a subinterval of $[\frac{1}{b} - a, \frac{1}{b}]$; thus (2.3) holds for $n = 1$ and $x \in]\alpha, \frac{1}{b}[$. Similarly, (2.3) holds for $n = -1$ and $x \in]-\frac{1}{b} - a, -a-\alpha[$. Note that $g(x - \frac{1}{b} + a) = 0$ if and only if $x = y_+ + \frac{1}{b} - a$. Let us for a moment assume that h is chosen on $[\frac{1}{b} - a, \alpha]$ as a bounded function such that h is continuous at $y = y_+ + \frac{1}{b} - a$ and

$$\lim_{y\to y_+} \left\{ h(y + (1-ab)\frac{n_+}{b}) R_{n_+}(y) \right\} \tag{2.4}$$

exists; letting $x = y + \frac{1}{b} - a$, this means that

$$\lim_{x \to y_+ + \frac{1}{b} - a} \left\{ \frac{h(x)}{g(x - \frac{1}{b} + a)} \right\}$$

exists. Then, defining h on $[\frac{1}{b}, a + \alpha]$ by

$$h(x+a) = \begin{cases} -\dfrac{g(x - \frac{1}{b})h(x)}{g(x - \frac{1}{b} + a)}, & x \in [\frac{1}{b} - a, \alpha] \setminus \{y_+ + \frac{1}{b} - a\}; \\ -\lim_{t \to y_+ + \frac{1}{b} - a} \left\{ \dfrac{h(t)}{g(t - \frac{1}{b} + a)} \right\} g(x - \frac{1}{b}), & x = y_+ + \frac{1}{b} - a, \end{cases}$$

(2.3) holds for $n = 1$ and $x \in [\frac{1}{b} - a, \alpha]$. Hence $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame if we can define h as a bounded function on $[-\alpha, \alpha]$ such that

(a) h is continuous at $y = y_+ + \frac{1}{b} - a$ and (2.4) holds;

(b) the duality condition (2.3) holds for $n = 0$ and $x \in [-a, 0]$, *i.e.*,

$$g(x)h(x) + g(x+a)h(x+a) = b, \; x \in [-a, 0]. \tag{2.5}$$

Let $\tilde{y}_+ := y_+ + (1 - ab)\frac{1}{b}$ and let $B_1 :=]y_+ - \epsilon, y_+ + \epsilon[\cup]\tilde{y}_+ - \epsilon, \tilde{y}_+ + \epsilon[\cup]\alpha - \epsilon, \alpha + \epsilon[$ and $B_2 :=] - \alpha - \epsilon, -\alpha + \epsilon[$, for an $\epsilon > 0$ chosen such that

(i) $|g(x)| \geq \delta > 0$ for $x \in (B_1 - a) \cup (B_2 + a)$ and some $\delta > 0$;

(ii) $B_1 \cap (B_2 + a) = \emptyset$.

Note that $g(x) \neq 0, \; x \in [\alpha - a, a - \alpha]$. By continuity of g, $\inf_{x \in [\alpha - a, a - \alpha]} |g(x)| > 0$. We define $h(x) := \frac{b}{g(x)}, \; x \in [\alpha - a, a - \alpha]$, which is thus a bounded function. Note that for $x \in [-a, -\alpha]$, we have $g(x) = 0$, and therefore

$$g(x)h(x) + g(x+a)h(x+a) = b. \tag{2.6}$$

Similarly, (2.6) holds for $x \in [\alpha - a, 0]$, *i.e.*, we have now verified (b) on the subinterval $[-a, -\alpha] \cup [\alpha - a, 0]$. We now put $h = 0$ on $B_1 \cap [a - \alpha, \alpha]$. Then h is continuous at $y = y_+ + \frac{1}{b} - a$ and $\lim_{y \to y_+} \left\{ h(y + (1 - ab)\frac{1}{b}) R_1(y) \right\} = 0$. Hence (a) holds. We define h on $(B_1 - a) \cap [-\alpha, \alpha - a]$ by $h(x) = \frac{b - g(x+a)h(x+a)}{g(x)} = \frac{b}{g(x)}$; thus h is bounded here by the choice of ϵ, and (b) holds on $(B_1 - a) \cap [-\alpha, \alpha - a]$. Similarly, put $h = 0$ on $B_2 \cap [-\alpha, \alpha - a]$ and define h on $(B_2 + a) \cap [a - \alpha, \alpha]$ by $h(x) = \frac{b}{g(x)}$; thus h is bounded, and (b) holds on $B_2 \cap [-\alpha, \alpha - a]$. We finally put $h = 0$ on $[-\alpha, \alpha - a] \setminus ((B_1 - a) \cup B_2)$. Note that the zeroset of g within $[-\alpha, \alpha]$ is $\{-\alpha, y_+, \alpha\}$, so $g(x) \neq 0$ for $x \in [a - \alpha, \alpha] \setminus (B_1 \cup (B_2 + a))$; using the continuity of g implies that $\inf_{x \in [a - \alpha, \alpha] \setminus (B_1 \cup (B_2 + a))} |g(x)| > 0$. We define $h(x) = \frac{b}{g(x)}$, $x \in [a-\alpha, \alpha] \setminus (B_1 \cup (B_2 + a))$; thus, we have now defined h everywhere as a bounded function, and (b) holds for $x \in [-\alpha, \alpha - a] \setminus ((B_1 - a) \cup B_2)$. This completes the proof of (b), and hence the proof of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ being a Gabor frame with a dual window supported on $[-2, 2]$. □

From Theorem 2.1 we can immediately extract the possible obstruction curves, i.e., the points (a, b) for which a given function $g \in V_\alpha$ might not generate a frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. Assume that $g \in V_\alpha$ satisfies the standing assumptions in Theorem 2.1 as well as the condition

$$|g(x)| + |g(x + a)| > 0, \ x \in [-a, 0]. \tag{2.7}$$

Then, if $\kappa \neq 0$, the possible obstructions take place on the curves determined by the equations

$$y_+ + (1 - ab)\frac{n_+}{b} - a = y_-, \tag{2.8}$$

$$y_- - (1 - ab)\frac{n_-}{b} + a = y_+, \tag{2.9}$$

$$y_+ + (1 - ab)\frac{n_+}{b} = y_- - (1 - ab)\frac{n_-}{b} + a. \tag{2.10}$$

for some y_+, y_-, n_+, n_- as in the theorem. The equations (2.8) and (2.9) both take the form

$$b = \frac{n}{y_- - y_+ + an + a} \tag{2.11}$$

for some $n \in \{1, 2, \cdots, \kappa\}$, while (2.10) means that

$$b = \frac{n_- + n_+}{y_- - y_+ + (n_- + n_+)a + a} \tag{2.12}$$

for some $n_-, n_+ \in \{1, 2, \cdots, \kappa\}$. Note that these curves only depend on the location of the zeros of the function $g \in V_\alpha$, not on the specific function.

Interestingly, the equations (2.11) and (2.12) show that for functions $g \in V_\alpha$ the obstructions take place on "hyperbolic curves:" this is similar to the result in [15], where Lyubarski and Nes showed that for any odd function in the Feichtinger algebra M^1, (in particular, the first order Hermite function) the points (a, b) for which $ab = 1 - 1/M = \frac{M-1}{M}$ for $M = 2, 3, \ldots$ do not belong to the frame set.

For functions $g \in V_\alpha$ with no zeroes in $]-\alpha, \alpha[$ the conditions in Theorem 2.1 are clearly satisfied, which yields the following:

Corollary 2.3 *Let $\alpha > 0$. Assume that g is a continuous function with* supp $g = [-\alpha, \alpha]$, *and that*

$$g(x) > 0, \quad x \in]-\alpha, \alpha[.$$

Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame whenever $\alpha \le a < 2\alpha$, $0 < b < 1/a$.

3 B-splines and a conjecture by Gröchenig

Let us now consider the B-splines B_N, $N \in \mathbb{N}$, defined recursively by

$$B_1 = \chi_{[-1/2,1/2]},\ B_{N+1} = B_N * B_1.$$

The frame properties of $\{E_{mb}T_{na}B_1\}_{m,n\in\mathbb{Z}}$ are well known (see the work by Janssen [12] and [3] by Dai and Sun which finally solved the so-called *abc*-problem), so we focus on the case $N \ge 2$, where B_N is a continuous function supported on $[-N/2, N/2]$. Furthermore the function B_N, $N \ge 2$, is strictly positive on the interval $]-N/2, N/2[$, so Corollary 2.3 implies that $\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ is a frame whenever $N/2 \le a < N$, $0 < b < 1/a$. Several other results about the frame set $\mathcal{F}(B_N)$ are known. We collect them here for easy reference:

Proposition 3.1 *Let $N \in \mathbb{N} \setminus \{1\}$, and consider $a, b > 0$ such that $ab < 1$. Then the following hold:*

(i) *$\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ is not a frame if $a \ge N$.*

(ii) *$\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ is not a frame if $b = 2, 3, \ldots$.*

(iii) *$\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ is a frame if $a < N$, $b \le 1/N$.*

(iv) *$\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ is a frame if there exists $k \in \mathbb{N}$ such that*

$$1/N < b < 2/N,\ N/2 \le ak < 1/b. \tag{3.1}$$

(v) *$\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ is a frame if $b \in \{1, \frac{1}{2}, \ldots, \frac{1}{N-1}\}$.*

(vi) *$\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ is a frame if $a = \frac{k}{p}$ for some $k = 1, \ldots, N-1$, $p \in \mathbb{N}$, and $b < 1/k$.*

Proof. The results in (i) and (iii) are classical. Also (ii) is a well known result, originally due to Del Prete [4] and rediscovered in [7]. For $k = 1$, the statement in (iv) is a consequence of Corollary 2.3. In general, if (3.1) holds for some $k \in \mathbb{N}\setminus\{1\}$, then this implies that $\{E_{mb}T_{nka}B_N\}_{m,n\in\mathbb{Z}}$ is a frame, and we infer that the larger system $\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ itself is a frame (because

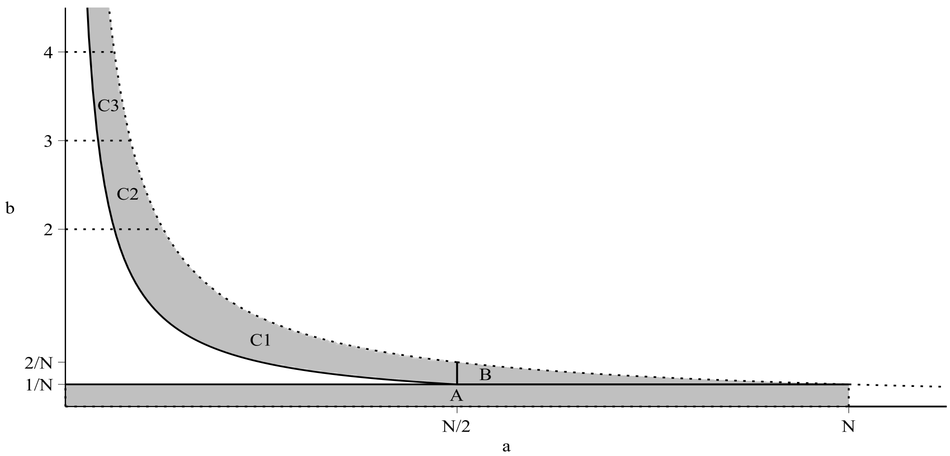

Figure 1: The set A belongs to the frame set for $B_N, N > 1$. Corollary 2.3 proves that B also belongs to the frame set (see also the introduction to Section 3); and by Proposition 3.2 the conjecture by Gröchenig is true if it can be verified in the regions $C1, C2, \ldots$.

the upper bound holds automatically). The result in (v) was recently proved by Kloos and Stöckler [13], who also proved (vi) for $p = 1$; The case of $p \in \mathbb{N}$ in (vi) yields an oversampling of the case $p = 1$, and therefore a frame. $\square$

Based on (i)–(iii) in Proposition 3.1 Gröchenig formulated a conjecture about the frame set $\mathcal{F}(B_N)$ in [6]. Basically it says that the frame set consists of all the points $(a, b) \in \mathbb{R}_+^2$ that avoids the known obstructions:

Conjecture For any $N \geq 2$,

$$\mathcal{F}(B_N) = \{(a, b) \in \mathbb{R}_+^2 \,\big|\, a < N,\, ab < 1, b \neq 2, 3, \ldots\}.$$

We will now show that the conjecture is true if we can prove the frame property in a certain " hyperbolic strip."

Proposition 3.2 *The conjecture is true if* $\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ *is a frame for all* $(a, b) \in \mathbb{R}_+^2$ *for which*

$$a < N/2,\ 1/2 \leq ab < 1,\ b \notin \{2, 3, \ldots\}. \tag{3.2}$$

Proof. To get a geometric understanding we refer to Figure 1. We note that Corollary 2.3 confirms the frame property in the region determined by the inequalities $N/2 < a < N, ab < 1$; furthermore the frame property is satisfied for $a < N, b \leq 1/N$ (i.e., the region A on Figure 1). Thus, it suffices to show that the parameter region determined by the inequalities

$$0 < ab < \frac{1}{2},\ \frac{1}{N} < b \notin \{2, 3, \ldots\}$$

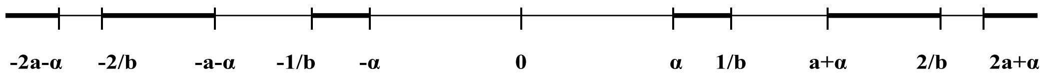

Figure 2: The figure shows the set where h is defined to vanish by (3.4), in the case $\kappa = 2$.

is contained in the frame set $\mathcal{F}(B_N)$ under the given assumption. Note that $\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for all $a, b > 0$, i.e., we only need to check the lower frame condition.

Since $0 < 2ab < 1$, choose the unique $M \in \mathbb{N}$ such that $\frac{1}{M+1} \le 2ab < \frac{1}{M}$. By splitting into the cases $2Ma < N/2$ and $2Ma \ge N/2$ it follows that the system $\{E_{mb}T_{n2Ma}B_N\}_{m,n\in\mathbb{Z}}$ is a frame; this clearly implies that $\{E_{mb}T_{na}B_N\}_{m,n\in\mathbb{Z}}$ satisfies the lower frame bound as well. $\square$

Appendix: Proof of Theorem 2.1

Let $M \in \mathbb{N}$, and assume that $\frac{M-1}{M} \le ab < \frac{M}{M+1}$. The starting point is the duality conditions by Ron & Shen [17, 16] and Janssen [9], which by an easy calculation implies that two real valued, bounded functions $g, h \in L^2(\mathbb{R})$ with supp $g \subseteq [-a, a]$, supp $h \subseteq [-aM, aM]$, generate dual frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ if and only if for $n = 0, \pm1, \pm2, \cdots, \pm(M-1)$ and *a.e.* $x \in [\frac{n}{b} - a, \frac{n}{b}]$,

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + a)h(x + a) = b\delta_{n,0}. \tag{3.3}$$

We will now consider a function $g \in V_\alpha$ that satisfies the conditions (i)–(iv) in Theorem 2.1. We will prove that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame by constructing a dual window h. In the following lemma, we use the insight gained from the proofs in [2] to define h on certain intervals, in such a way that (3.3) is satisfied for some of the relevant values of n and on certain intervals. After that the subsequent lemma states conditions that yields a definition of h on the remaining parts of the real line in such a way that all the duality conditions are satisfied.

Lemma 3.3 *Let $\alpha, a, b > 0$ be given such that $\alpha \le a < 2\alpha$ and $ab \in [\frac{M-1}{M}, \frac{M}{M+1}[$ for some $M \in \mathbb{N} \setminus \{1\}$. Assume that*

$$h(x) = 0, \quad x \notin -\left(\bigcup_{k=1}^{\kappa}[\frac{k}{b}, ak + \alpha]\right) \cup [-\alpha, \alpha] \cup \bigcup_{k=1}^{\kappa}[\frac{k}{b}, ak + \alpha]. \tag{3.4}$$

Then the following hold:

(a) $h(x) = h(x+a) = 0$ *for* $n = 1, \cdots, \kappa$ *and* $x \in]\alpha + a(n-1), \frac{n}{b}[$, *and for* $n = -1, \cdots, -\kappa$ *and* $x \in]\frac{n}{b} - a, an - \alpha[$;

(b) $h(x) = h(x+a) = 0$ *for* $n = \pm(\kappa+1), \cdots, \pm(M-1)$ *and* $x \in [\frac{n}{b} - a, \frac{n}{b}]$.

Proof. Note that for $n = 1, 2, \ldots, M-1$, $ab \geq \frac{n}{n+1}$; thus, $\frac{n}{b} - a \leq an < \frac{n}{b}$.
(a): For $1 \leq n \leq \kappa$, we note that the statement in (a) only involves the function values of h for $x \in]\alpha + a(n-1), n/b[$ and $x + a \in]\alpha + an, n/b + a[$. Since $(]\alpha + a(n-1), n/b[\cup]\alpha + an, n/b + a[) \cap \mathrm{supp}\, h = \emptyset$, (a) holds for $1 \leq n \leq \kappa$. Similarly, (a) holds for $-\kappa \leq n \leq -1$.
(b): For $\kappa + 1 \leq n \leq M-1$, the statement in (b) only involves the values of h for $x \in [\frac{n}{b} - a, \frac{n}{b}]$ and $x + a \in [\frac{n}{b}, \frac{n}{b} + a]$; by the definition of κ, we have $b\alpha < (\kappa+1)(1-ab)$, *i.e.*, $a\kappa + \alpha < \frac{\kappa+1}{b} - a$; thus $\left([\frac{n}{b} - a, \frac{n}{b} + a]\right) \cap \mathrm{supp}\, h = \emptyset$. Hence (3.3) holds for $\kappa + 1 \leq n \leq M - 1$. Similarly, (b) holds for $-M + 1 \leq n \leq -\kappa - 1$. □

Note that condition (b) in Lemma 3.3 is empty if $\kappa = 0$; condition (c) is empty if $\kappa = M - 1$.

By Lemma 3.3, we see that (3.3) holds for $n = 1, \cdots, \kappa$ and $x \in]\alpha + a(n-1), \frac{n}{b}[$, and for $n = -1, \cdots, -\kappa$ and $x \in]\frac{n}{b} - a, an - \alpha[$. Similarly, (3.3) holds for $n = \pm(\kappa+1), \cdots, \pm(M-1)$ and $x \in [\frac{n}{b} - a, \frac{n}{b}]$. What remains is to show that we can define h on the set

$$-\left(\bigcup_{k=1}^{\kappa} [\frac{k}{b}, ak + \alpha]\right) \cup [-\alpha, \alpha] \cup \bigcup_{k=1}^{\kappa} [\frac{k}{b}, ak + \alpha]$$

such that (3.3) holds for $n = 1, \cdots, \kappa$ and $x \in [\frac{n}{b} - a, \alpha + a(n-1)]$, and for $n = -1, \cdots, -\kappa$ and $x \in [an - \alpha, \frac{n}{b}]$, as well as for $n = 0$ and $x \in [-a, 0]$. The following lemma states sufficient conditions for the first of these requirements to be satisfied. The result is a minor adaption of Lemma 3.3 in [2], so the proof is omitted.

Lemma 3.4 *Let* $\alpha, a, b > 0$ *be given such that* $\alpha \leq a < 2\alpha$ *and* $ab \in [\frac{M-1}{M}, \frac{M}{M+1}[$ *for some* $M \in \mathbb{N} \setminus \{1\}$. *Let* $g \in V_\alpha$, *and assume that* $g(x) \neq 0$ *for* $x \in [\alpha - a, a - \alpha]$. *Assume further that* $\kappa \neq 0$ *and that* h *is chosen on* $[-\alpha, \alpha]$ *as a bounded function such that the following conditions hold:*

(1) *If there exist* $n_+ \in \{1, 2, \cdots, \kappa\}$ *and* $y_+ \in]a - \alpha, \alpha - (1 - ab)\frac{n_+}{b}]$ *such that* $g(y_+) = 0$ *and* $\lim_{y \to y_+} |R_{n_+}(y)| = \infty$, *then* h *is continuous at* $y = y_+ + (1 - ab)\frac{n_+}{b}$ *and the limit*

$$\lim_{y \to y_+} \left\{ h(y + (1 - ab)\frac{n_+}{b}) R_{n_+}(y) \right\} \tag{3.5}$$

exists;

(2) *If there exist* $n_- \in \{1, 2, \cdots, \kappa\}$ *and* $y_- \in [-\alpha + (1-ab)\frac{n_-}{b}, \alpha - a]$ *such that* $g(y_-) = 0$ *and* $\lim_{y \to y_-} |L_{n_-}(y)| = \infty$, *then* h *is continuous at* $y = y_- - (1-ab)\frac{n_-}{b}$ *and the limit*

$$\lim_{y \to y_-} \left\{ h(y - (1-ab)\frac{n_-}{b}) L_{n_-}(y) \right\} \tag{3.6}$$

exists.

Then the function h *can be defined on the interval* $-(\bigcup_{k=1}^{\kappa}[\frac{k}{b}, ak+\alpha]) \cup \bigcup_{k=1}^{\kappa}[\frac{k}{b}, ak+\alpha]$ *such that the duality condition* (3.3) *holds for* $n = 1, \cdots, \kappa$ *and* $x \in [\frac{n}{b} - a, \alpha + a(n-1)]$, *as well as for* $n = -1, \cdots, -\kappa$ *and* $x \in [an - \alpha, \frac{n}{b}]$; *the function* h *is bounded, and the values of* h *outside* $-(\bigcup_{k=1}^{\kappa}[\frac{k}{b}, ak+\alpha]) \cup [-\alpha, \alpha] \cup \bigcup_{k=1}^{\kappa}[\frac{k}{b}, ak+\alpha]$ *are irrelevant.*

Remark 3.5 In Lemma 3.4, if y_+ is the end point of the interval $]a - \alpha, \alpha - (1-ab)\frac{n_+}{b}]$, *i.e.*, $y_+ = \alpha - (1-ab)\frac{n_+}{b}$, the limit $\lim_{y \to y_+}$ in (3.5) should be understood as the limit from the left; similarly, if $y_- = -\alpha + (1-ab)\frac{n_-}{b}$, the limit $\lim_{y \to y_+}$ in (3.6) should be understood as the limit from the right. $\square$

We can now complete the proof of the sufficiency in Theorem 2.1:

Proof of Theorem 2.1: Assume that the conditions (i)–(iv) in Theorem 2.1 hold. Note that $g(x) = 0,\ x \in [-a, -\alpha] \cup [\alpha, a]$, since $g \in V_\alpha$. This together with condition (i) in Theorem 2.1 implies that

$$g(x) \neq 0,\ x \in [\alpha - a, a - \alpha]. \tag{3.7}$$

Following (3.4), let

$$h(x) := 0, \quad x \notin -\left(\bigcup_{k=1}^{\kappa} [\frac{k}{b}, ak+\alpha] \right) \cup [-\alpha, \alpha] \cup \bigcup_{k=1}^{\kappa} [\frac{k}{b}, ak+\alpha].$$

Via Lemma 3.4 and the comment just before the lemma, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame if we can define h as a bounded function on $[-\alpha, \alpha]$ in such a way that

(a) the conditions in Lemma 3.4 (1) and (2) hold;

(b) the duality condition (3.3) holds for $n = 0$ and $x \in [-a, 0]$, *i.e.*,

$$g(x)h(x) + g(x+a)h(x+a) = b,\ x \in [-a, 0]. \tag{3.8}$$

We will split the definition of h on $[-\alpha, \alpha]$ into several intervals. In fact, we will first define h on $[\alpha - a, a - \alpha]$ and then on small balls around certain shifts of the zeros. First, we need some notation. For $m, n = 0, 1, \cdots, \kappa$, we define the sets Y_n and W_m by

$$Y_0 = \{y_{0,i} \in]a - \alpha, \alpha] \ : g(y_{0,i}) = 0\}_{i=1,2,\cdots,r_0}$$
$$Y_n = \{y_{n,i} \in]a - \alpha, \alpha - (1 - ab)\frac{n}{b}] \ : g(y_{n,i}) = 0 \text{ and } \lim_{y \to y_{n,i}} |R_n(y)| = \infty\}_{i=1,2,\cdots,r_n}$$

and

$$W_0 = \{w_{0,j} \in [-\alpha, \alpha - a[\ : g(w_{0,j}) = 0\}_{j=1,2,\cdots,l_0}$$
$$W_m = \{w_{m,j} \in [-\alpha + (1 - ab)\frac{m}{b}, \alpha - a[\ : g(w_{m,j}) = 0 \text{ and } \lim_{y \to w_{m,j}} |L_m(y)| = \infty\}_{j=1,2,\cdots,l_m}$$

where r_n and l_m are the cardinalities of Y_n and W_m, respectively. In words: since $g(x) \neq 0$ for $x \in [\alpha - a, a - \alpha]$, the sets Y_0 and W_0 yield enumerations of the zeros for g within $[-\alpha, \alpha]$, split into the positive, respectively, negative part; the sets Y_n and $W_n, n \geq 1$, yield enumerations of selected zeros within certain subsets of $[-\alpha, \alpha]$.

We denote the open interval of radius $r > 0$ centered at x by $B(x; r) =]x - r, x + r[$. For $y_{n,i} \in Y_n$, $w_{m,j} \in W_m$ for $n, m = 0, 1, \cdots, \kappa$, let $\tilde{y}_{n,i} := y_{n,i} + (1 - ab)\frac{n}{b}$, $\hat{w}_{m,j} := w_{m,j} - (1 - ab)\frac{m}{b}$. If $n, m \geq 1$, then by the conditions (ii), (iii) and (iv) in Theorem 2.1(3), we have

$$g(\tilde{y}_{n,i} - a) \neq 0 \neq g(\hat{w}_{m,j} + a), \text{ and } \tilde{y}_{n,i} \neq \hat{w}_{m,j} + a. \tag{3.9}$$

Note that $g(\tilde{y}_{0,i}) = g(\hat{w}_{0,j}) = 0$. Then we also have $\tilde{y}_{0,i} \neq \hat{w}_{m,j} + a$, $\tilde{y}_{n,i} - a \neq \hat{w}_{0,j}$ for $m, n \geq 1$, and $g(\tilde{y}_{0,i} - a) \neq 0 \neq g(\hat{w}_{0,j} + a)$ by the condition (i) in Theorem 2.1; thus, (3.9) actually holds for all $m, n = 0, 1, \cdots, \kappa$. Then we can choose $\epsilon > 0$ so that

(i) $|g(x)| \geq \delta > 0$ for $x \in B(\tilde{y}_{n,i} - a; \epsilon) \cup B(\hat{w}_{m,j} + a; \epsilon)$ and some $\delta > 0$;

(ii) For $m, n = 0, 1, \cdots, \kappa$, and $i = 1, 2, \cdots, r_n, j = 1, 2, \cdots, l_m$,

$$B(\tilde{y}_{n,i}; \epsilon) \cap B(\hat{w}_{m,j} + a; \epsilon) = \emptyset. \tag{3.10}$$

Definition of h on $[\alpha - a, a - \alpha]$: By (3.7) and continuity of g, $\inf_{x \in [\alpha - a, a - \alpha]} |g(x)| > 0$. We define $h(x) := \frac{b}{g(x)}$, $x \in [\alpha - a, a - \alpha]$, which is thus a bounded function. Note that for $x \in [-a, -\alpha]$, we have $g(x) = 0$, and therefore

$$g(x)h(x) + g(x + a)h(x + a) = b. \tag{3.11}$$

Similarly, (3.11) holds for $x \in [\alpha - a, 0]$, *i.e.*, we have now verified (b) on the subinterval $[-a, -\alpha] \cup [\alpha - a, 0]$.

Definition of h on $B(\tilde{y}_{n,i};\epsilon)\cap[a-\alpha,\alpha]$: On this interval, put $h=0$. If $1\le n\le\kappa$, then, h is continuous at $y=\tilde{y}_{n,i}$ and

$$\lim_{y\to y_{n,i}}\left\{h(y+(1-ab)\frac{n}{b})R_n(y)\right\}=0. \tag{3.12}$$

Hence the condition in Lemma 3.4 (1) holds.
Definition of h on $B(\tilde{y}_{n,i}-a;\epsilon)\cap[-\alpha,\alpha-a]$: We define h on this set by $h(x)=\frac{b-g(x+a)h(x+a)}{g(x)}=\frac{b}{g(x)}$; thus h is bounded here by the choice of ϵ, and (b) holds on $B(\tilde{y}_{n,i}-a;\epsilon)\cap[-\alpha,\alpha-a]$.
Definition of h on $B(\hat{w}_{m,j};\epsilon)\cap[-\alpha,\alpha-a]$: On this interval, put $h=0$. If $1\le m\le\kappa$, then h is continuous at $y=\hat{w}_{m,j}$ and

$$\lim_{y\to w_{m,j}}\left\{h(y-(1-ab)\frac{m}{b})L_m(y)\right\}=0. \tag{3.13}$$

Hence the condition in Lemma 3.4 (2) holds, *i.e.*, we have now completed the proof of (a).
Definition of h on $B(\hat{w}_{m,j}+a;\epsilon)\cap[a-\alpha,\alpha]$: We define h on this set by $h(x)=\frac{b-g(x-a)h(x-a)}{g(x)}=\frac{b}{g(x)}$; thus h is bounded here by the choice of ϵ and (3.10), and (b) holds on $B(\hat{w}_{m,j};\epsilon)\cap[-\alpha,\alpha-a]$.

To summarize all these, let $B_+:=\cup_{n=0}^{\kappa}\cup_{i=1}^{r_n}\left(B(\tilde{y}_{n,i};\epsilon)\cap[a-\alpha,\alpha]\right)$, and $B_-:=\cup_{m=0}^{\kappa}\cup_{j=1}^{l_m}\left(B(\hat{w}_{m,j};\epsilon)\cap[-\alpha,\alpha-a]\right)$. We have defined h as a bounded function on $B:=[\alpha-a,a-\alpha]\cup B_+\cup(B_+-a)\cup B_-\cup(B_-+a)$, and (b) holds on

$$[-a,-\alpha]\cup[\alpha-a,0]\cup(B_+-a)\cup B_-=[-a,-\alpha]\cup(B\cap[-a,0]). \tag{3.14}$$

Definition of h on $[-\alpha,\alpha]\setminus B$: Put $h=0$ on $([-\alpha,\alpha]\setminus B)\cap[-a,0]$. Note that the zeroset of g within $[-\alpha,\alpha]$ consists of Y_0 and W_0, so $g(x)\neq 0$ for $x\in\overline{[-\alpha,\alpha]\setminus B}$; using the continuity of g implies that $\inf_{x\in[-\alpha,\alpha]\setminus B}|g(x)|>0$. We define $h(x)=\frac{b}{g(x)},\ x\in([-\alpha,\alpha]\setminus B)\cap[0,a]$; thus, we have now defined h everywhere as a bounded function, and we just need to complete the proof of (b). Since we have proved (b) on the set in (3.14), we just need to verify (b) on the set $]-\alpha,0]\setminus(B\cap[-a,0])$. Note that h vanishes on this set and that

$$\begin{aligned}]-\alpha,0]\setminus(B\cap[-a,0]) &= ([-\alpha,\alpha]\setminus B)\cap[-a,0]\\ &=]-\alpha,\alpha-a[\setminus((B_+-a)\cup B_-)\\ &= (([-\alpha,\alpha]\setminus B)\cap[0,a])-a,\end{aligned}$$

where we used that $-\alpha\in B_-,\ \alpha\in B_+$. Thus, by the definition of h on $([-\alpha,\alpha]\setminus B)\cap[0,a]$ (b) holds on $]-\alpha,0]\setminus(B\cap[-a,0])$, as desired. □

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