Extensions of Bessel sequences to dual pairs of frames*

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Abstract

Tight frames in Hilbert spaces have been studied intensively for the past years. In this paper we demonstrate that it often is an advantage to use pairs of dual frames rather than tight frames. We show that in any separable Hilbert space, any pairs of Bessel sequences can be extended to a pair of dual frames. If the given Bessel sequences are Gabor systems in $L^2(\mathbb{R})$, the extension can be chosen to have Gabor structure as well. We also show that if the generators of the given Gabor Bessel sequences are compactly supported, we can choose the generators of the added Gabor systems to be compactly supported as well. This is a significant improvement compared to the extension of a Bessel sequence to a tight frame, where the added generator only can be compactly supported in some special cases. We also analyze the wavelet case, and find sufficient conditions under which a pair of wavelet systems can be extended to a pair of dual frames.

Keywords Bessel sequences, Dual frame pairs, Gabor frames, wavelet frames
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1 Introduction

Extension principles have a long history in frame theory. In 1997 Ron and Shen introduced the unitary extension principle, which allows certain wavelet systems to be extended to tight wavelet frames in $L^2(\mathbb{R})$ by adding extra generators [13], [14]. In [2], Casazza and Leonhard showed that any Bessel sequence in a finite-dimensional space can be extended to a tight frame, a result that was later extended to the infinite-dimensional case by Li and Sun [11]. The purpose of this note is to consider extension of Bessel sequences to more general dual frame pairs. The advantage of this is that dual frame pairs is a more flexible tool than tight frames. Therefore the extension to a dual frame pair might be computationally more efficient, and we might obtain properties that are impossible if we extend to a tight frame. We will see theoretical results and examples of both types.

As starting point we will show in Section 2 that any pair of Bessel sequences in a separable Hilbert space can be extended to a pair of dual frames. In contrast with the known extension of a Bessel sequence to a tight frame, the procedure does not involve calculation of the square root of an operator. Furthermore, we give simple examples where the extension to a dual frame pair is computationally more efficient than the extension to a tight frame.

In Section 3 we consider the case of Bessel sequences in $L^2(\mathbb{R})$ having Gabor structure. We show that if the given Bessel sequences have Gabor structure, it is always possible to extend to a dual frame pair by adding one extra Gabor system to each Bessel sequence. We also show that if the generators for the given Bessel sequences are compactly supported, the generators for the resulting dual frame pair can be chosen to be compactly supported as well. This is a serious improvement compared to the known extension to a tight frame, where such a result only holds if the generator of the given Bessel sequence has sufficiently small support.

We also analyze the corresponding problem for wavelet systems and find sufficient conditions under which a pair of wavelet Bessel sequences can be extended to a pair of dual wavelet frames. Our results are complementary to the classical extension principles in wavelet analysis: in fact, our results require the Fourier transform of the given generators to be compactly supported, while the unitary extension principle and its subsequent generalizations usually have been applied for the construction of wavelet systems where the generators are compactly supported.

In the rest of the introduction we review the needed facts from frame theory. Let $\mathcal{H}$ denote a separable Hilbert space. A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\,||f||^2, \ \forall f \in \mathcal{H}. \tag{1.1}$$

The sequence $\{f_i\}_{i\in I}$ is a *Bessel sequence* if at least the upper bound in (1.1) is satisfied. The number B is called an *upper frame bound*, and the smallest possible value for B is called the *optimal bound*. A frame is *tight* if we can choose $A = B$ in (1.1).

For any frame $\{f_i\}_{i\in I}$ there exists at least one *dual frame,* i.e., a frame $\{g_i\}_{i\in I}$ for which

$$f = \sum_{i\in I} \langle f, f_i\rangle g_i, \ \forall f \in \mathcal{H}. \tag{1.2}$$

If $\{f_i\}_{i\in I}$ is a tight frame with $A = B = 1$, one can take $g_i = f_i$ and we obtain the representation

$$f = \sum_{i\in I} \langle f, f_i\rangle f_i, \ \forall f \in \mathcal{H}. \tag{1.3}$$

The desire of obtaining (1.3) immediately motivates the problems considered in [13], [14], [2], [11] in various settings: if $\{f_i\}_{i\in I}$ is a Bessel sequence with bound $B \le 1$, how can we find a sequence $\{p_j\}_{i\in J}$ such that$\{f_i\}_{i\in I} \cup \{p_j\}_{i\in J}$ is a tight frame with bound $A = 1$, i.e., such that

$$f = \sum_{i\in I} \langle f, f_i\rangle f_i + \sum_{j\in J} \langle f, p_j\rangle p_j, \ \forall f \in \mathcal{H}? \tag{1.4}$$

In this paper we aim at the more general results of the type in (1.2). Thus, given Bessel sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$, we ask for the existence of Bessel sequences $\{p_j\}_{i\in J}$ and $\{q_j\}_{i\in J}$ such that $\{f_i\}_{i\in I} \cup \{p_j\}_{i\in J}$ and $\{g_i\}_{i\in I} \cup \{q_j\}_{i\in J}$ are dual frames, i.e., such that

$$f = \sum_{i\in I} \langle f, f_i\rangle g_i + \sum_{j\in J} \langle f, p_j\rangle q_j, \ \forall f \in \mathcal{H}. \tag{1.5}$$

Note that even in the case where $g_i = f_i$, it is an advantage to consider the more general extension problem in (1.5) rather than (1.4). For illustration of this point, see the comment after Theorem 2.1, Example 2.2, and, in the Gabor setting, Theorem 3.1 and Example 3.2.

For more information on general frames we refer to the books [5], [15], [3].

2 Extensions of Bessel sequences in general Hilbert spaces

Given any Bessel sequence $\{f_i\}_{i\in I}$ in a separable Hilbert space $\mathcal{H}$, it was shown by Li and Sun [11] that there exists a sequence $\{p_j\}_{i\in J}$ in $\mathcal{H}$ such that $\{f_i\}_{i\in I}\cup\{p_j\}_{i\in J}$ is a tight frame for $\mathcal{H}$. We first extend this result by showing that any pair of Bessel sequences can be extended to a pair of dual frames:

Proposition 2.1 *Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be Bessel sequences in a Hilbert space $\mathcal{H}$. Then there exist Bessel sequences $\{p_j\}_{i\in J}$ and $\{q_j\}_{i\in J}$ in $\mathcal{H}$ such that $\{f_i\}_{i\in I}\cup\{p_j\}_{i\in J}$ and $\{g_i\}_{i\in I}\cup\{q_j\}_{i\in J}$ form a pair of dual frames for $\mathcal{H}$.*

Proof. Let T and U denote the preframe operators (or synthesis operators) for $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$, respectively, i.e.,

$$T, U : \ell^2(I) \to \mathcal{H},\ T\{c_i\}_{i\in I} = \sum_{i\in I} c_i f_i,\ \ U\{c_i\}_{i\in I} = \sum_{i\in I} c_i g_i.$$

Then

$$UT^* f = \sum_{i\in I} \langle f, f_i\rangle g_i,\ \forall f \in \mathcal{H}.$$

Consider the bounded operator $\Phi := I - UT^*$, and let $\{a_j\}_{i\in J}, \{b_j\}_{i\in J}$ denote any pair of dual frames for $\mathcal{H}$. Then

$$\Phi f = \sum_{j\in J} \langle \Phi f, a_j\rangle b_j = \sum_{j\in J} \langle f, \Phi^* a_j\rangle b_j,\ \forall f \in \mathcal{H}.$$

Thus,

$$f = UT^* f + \Phi f = \sum_{i\in I} \langle f, f_i\rangle g_i + \sum_{j\in J} \langle f, \Phi^* a_j\rangle b_j,\ \forall f \in \mathcal{H}. \tag{2.1}$$

The sequences $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$, and $\{b_j\}_{j\in J}$ are Bessel sequences by definition, and it is easy to verify that $\{\Phi^* a_j\}_{j\in J}$ is a Bessel sequence as well. Thus, (2.1) implies by [12] or Lemma 5.7.1 in [3] that $\{f_i\}_{i\in I} \cup \{\Phi^* a_j\}_{j\in J}$ and $\{g_i\}_{i\in I} \cup \{b_j\}_{i\in J}$ are dual frames for $\mathcal{H}$. $\square$

Note that the construction of the sequence $\{p_j\}_{i\in J}$ in Proposition 2.1 only involves the operator $\Phi^* = I - TU^*$. Thus, the practical computation is easier than the extension to a tight frame in [11], which involves computation of the square root of a bounded operator.

In the special case where $\{f_i\}_{i\in I} = \{g_i\}_{i\in I}$, Proposition 2.1 follows from the mentioned result by Li and Sun. But even in that case it is an advantage to consider extensions to dual frame pairs rather than to a tight frames. This is demonstrated in the context of Gabor systems in Section 3. For now, we just present a simple example which shows that extension to a dual frame pair might be significantly cheaper than extension to a tight frame.

Example 2.2 Let $\{e_j\}_{j=1}^{10}$ be an orthonormal basis for $\mathbb{C}^{10}$ and consider the frame

$$\{f_j\}_{j=1}^{10} := \{2e_1\} \cup \{e_j\}_{j=2}^{10}.$$

The frame $\{f_j\}_{j=1}^{10}$ is not tight, but we can make it tight by adding the vectors $\{g_j\}_{j=2}^{10} = \{\sqrt{3}e_j\}_{j=2}^{10}$. This is the optimal extension, in the sense that 9 is the *minimal* number of vectors that needs to be added in order to obtain a tight frame. On the other hand, a pair of dual frames can be obtained by adding just one element. In fact, the two sequences

$$\{f_j\}_{j=1}^{10} \cup \{-3e_1\} \quad \text{and} \quad \{f_j\}_{j=1}^{10} \cup \{e_1\}$$

form dual frames in $\mathbb{C}^{10}$. $\square$

The construction in Example 2.2 can clearly be extended to any dimension. Similar examples can be made for normalized frames, e.g., for the frame

$$\{f_j\}_{j=1}^{10} := \{\frac{e_1 + e_2}{\sqrt{2}}\} \cup \{e_j\}_{j=2}^{10}.$$

3 Extensions of Gabor Bessel sequences

If the given Bessel sequences have a special structure, it is interesting to know if we can keep the same structure for the extension to a dual frame pair. We first consider Gabor systems. Recall that a *Gabor system* in $L^2(\mathbb{R})$ has the form $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ for some parameters $a, b > 0$ and a given function $g \in L^2(\mathbb{R})$. Using the *translation operators* $T_af(x) := f(x-a), a \in \mathbb{R}$, and the *modulation operators* $E_bf(x) := e^{2\pi ibx}f(x), b \in \mathbb{R}$, both acting on $L^2(\mathbb{R})$, we will denote a Gabor system by $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. For more information about Gabor systems and their applications we refer to the book [9] as well as the collection of research articles in [7] and [8].

We will now show that any pair of Gabor Bessel sequences can be extended to a pair of dual frames with Gabor structure. We also prove that we can choose the generators of the added Gabor systems to be compactly supported if the given generators are compactly supported. This is a significant improvement compared to the extension of a Bessel sequence $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}}$ to a tight frame reported in [11]: in that case the compact support of the added generator is only guaranteed if the given generator g_1 has sufficiently small support, i.e., $|\text{supp}\, g_1| \leq 1/b$.

Theorem 3.1 *Let $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$, and assume that $ab \leq 1$. Then the following hold:*

(i) There exist Gabor systems $\{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h_2\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ such that

$$\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}} \text{ and } \{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}h_2\}_{m,n\in\mathbb{Z}}$$

form a pair of dual frames for $L^2(\mathbb{R})$.

(ii) If the functions g_1 and h_1 are compactly supported, the functions g_2 and h_2 can be chosen to be compactly supported as well.

(iii) Assume that $ab < 1$ and that the functions g_1 and h_1 are compactly supported and C^∞. Then the g_2 and h_2 can be chosen to be compactly supported and C^∞ as well.

Proof. Let T and U denote the preframe operators for $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}}$, respectively. Then

$$UT^*f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g_1\rangle E_{mb}T_{na}h_1.$$

Consider the operator $\Phi := I - UT^*$, and let $\{E_{mb}T_{na}r_1\}_{m,n\in\mathbb{Z}}$, $\{E_{mb}T_{na}r_2\}_{m,n\in\mathbb{Z}}$ denote any pair of dual frames for $L^2(\mathbb{R})$ (see Appendix A). By the proof of Proposition 2.1, $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{\Phi^* E_{mb}T_{na}r_1\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}r_2\}_{m,n\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$. Note that $\Phi^* = I - TU^*$, i.e.,

$$\Phi^* f = f - \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h_1\rangle E_{mb}T_{na}g_1. \tag{3.1}$$

A standard argument (as in Lemma 9.3.1 in [3]) shows that Φ^* commutes with all the time-frequency shift operators $E_{mb}T_{na}$. Thus we conclude that

$$\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}\Phi^* r_1\}_{m,n\in\mathbb{Z}} \text{ and } \{E_{mb}T_{na}h_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}r_2\}_{m,n\in\mathbb{Z}}$$

are dual frames for $L^2(\mathbb{R})$. This proves (i).

We now prove (ii). Take the functions r_1 and r_2 in the proof of (i) to be compactly supported (see Appendix A). Then we just need to show that the function $g_2 = \Phi^* r_1$ is compactly supported. Due to the compact support of the functions r_1 and h_1 there exists a number N such that

$$\langle r_1, E_{mb}T_{na}h_1\rangle = 0, \ \forall m \in \mathbb{Z} \text{ if } n \notin [-N, N].$$

Thus, by (3.1),

$$\begin{aligned} \Phi^* r_1 &= r_1 - \sum_{m,n\in\mathbb{Z}} \langle r_1, E_{mb}T_{na}h_1\rangle E_{mb}T_{na}g_1 \\ &= r_1 - \sum_{n=-N}^{N} \sum_{m\in\mathbb{Z}} \langle r_1, E_{mb}T_{na}h_1\rangle E_{mb}T_{na}g_1, \end{aligned} \tag{3.2}$$

which is clearly compactly supported.

For the proof of (iii), it suffices to show that $g_2 = \Phi^* r_1 \in C^\infty(\mathbb{R})$. For $k \in \mathbb{N}$, define a norm $||\cdot||_k : C^\infty \to \mathbb{R}$ by

$$||f||_k := \max_{0\le r\le k} ||f^{(r)}||_\infty.$$

Choose the functions r_1, r_2 in the proof of (ii) to be $C^\infty(\mathbb{R})$ (see Appendix A). We first estimate $|\langle r_1, E_{mb}T_{na}h_1\rangle|$. Choose L so that

$$\bigcup_{n=-N}^{N} \operatorname{supp}\left\{r_1(\cdot)\overline{h}_1(\cdot - na)\right\} \subset [-L, L].$$

For any $k \in \{0, 1, 2, \dots\}$, integration by parts implies

$$\begin{aligned}
\langle r_1, E_{mb}T_{na}h_1\rangle &= \int_{-\infty}^{\infty} e^{-2\pi imbx}(r_1(x)\overline{h}_1(x-na))dx \\
&= \frac{1}{2\pi imb}\int_{-\infty}^{\infty} e^{-2\pi imbx}(r_1(x)\overline{h}_1(x-na))'dx \\
&= \cdots \\
&= \frac{1}{(2\pi imb)^k}\int_{-\infty}^{\infty} e^{-2\pi imbx}(r_1T_{na}\overline{h}_1)^{(k)}(x)dx.
\end{aligned}$$

By Leibnitz formula for the k-derivative of a product, we have

$$(r_1T_{na}\overline{h}_1)^{(k)}(x) = \sum_{r=0}^{k}\binom{k}{r} r_1^{(r)}(x)\overline{h}_1^{(k-r)}(x-na).$$

Thus,

$$\begin{aligned}
||(r_1T_{na}\overline{h}_1)^{(k)}||_\infty &\leq \sum_{r=0}^{k}\binom{k}{r}||r_1^{(r)}||_\infty||\overline{h}_1^{(k-r)}(\cdot - na)||_\infty \\
&\leq \sum_{r=0}^{k}\binom{k}{r}||r_1||_k||h_1||_k \\
&= 2^k||r_1||_k||h_1||_k.
\end{aligned}$$

Hence, for any $k \in \{0, 1, \dots\}$,

$$|\langle r_1, E_{mb}T_{na}h_1\rangle| \leq \frac{2L}{(\pi mb)^k}||r_1||_k||h_1||_k. \tag{3.3}$$

We now estimate the derivative $(E_{mb}T_{na}g_1)^{(\ell)}(x),\ \ell \in \mathbb{N}:$

$$\begin{aligned}
\left|(E_{mb}T_{na}g_1)^{(\ell)}(x)\right| &= \left|\sum_{r=0}^{\ell}\binom{\ell}{r}(e^{2\pi imbx})^{(r)}g_1^{(\ell-r)}(x-na)\right| \\
&= \left|\sum_{r=0}^{\ell}\binom{\ell}{r}(2\pi imb)^r e^{2\pi imbx}g_1^{(\ell-r)}(x-na)\right| \\
&\leq \sum_{r=0}^{\ell}\binom{\ell}{r}(2\pi mb)^r||g_1||_\ell \\
&= (2\pi mb+1)^\ell\,||g_1||_\ell. \qquad (3.4)
\end{aligned}$$

Now, fix $\ell_0 \in \mathbb{N}$ and choose $k_0 \in \mathbb{N}$ so that $k_0 - \ell_0 > 1$. Using (3.3) with $k = k_0$ and (3.4), we have

$$\begin{aligned} & \sum_{n=-N}^{N} \sum_{m \in \mathbb{Z} \setminus \{0\}} || \left(\langle r_1, E_{mb} T_{na} h_1 \rangle E_{mb} T_{na} g_1 \right)^{(\ell_0)} ||_\infty \\ \leq & \left\{ \sum_{n=-N}^{N} \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{2L}{(\pi m b)^{k_0}} (2\pi m b + 1)^{\ell_0} \right\} ||r_1||_{k_0} ||h_1||_{k_0} ||g_1||_{\ell_0} \\ < & \ \infty, \end{aligned}$$

since $k_0 - \ell_0 > 1$. By Weierstrass M-test, the infinite series

$$\sum_{n=-N}^{N} \sum_{m \in \mathbb{Z} \setminus \{0\}} \left(\langle r_1, E_{mb} T_{na} h_1 \rangle E_{mb} T_{na} g_1 \right)^{(\ell_0)} (x)$$

converges uniformly for any $\ell_0 \in \mathbb{N}$. This implies that the function

$$\sum_{n=-N}^{N} \sum_{m \in \mathbb{Z} \setminus \{0\}} \langle r_1, E_{mb} T_{na} h_1 \rangle E_{mb} T_{na} g_1$$

is infinitely often differentiable, and that for any $\ell_0 \in \mathbb{N}$,

$$\begin{aligned} & \left(\sum_{n=-N}^{N} \sum_{m \in \mathbb{Z} \setminus \{0\}} \langle r_1, E_{mb} T_{na} h_1 \rangle E_{mb} T_{na} g_1 \right)^{(\ell_0)} \\ = & \sum_{n=-N}^{N} \sum_{m \in \mathbb{Z} \setminus \{0\}} \left(\langle r_1, E_{mb} T_{na} h_1 \rangle E_{mb} T_{na} g_1 \right)^{(\ell_0)}. \end{aligned}$$

Using the expression in (3.2), we deduce that $\Phi^* r_1 \in C^\infty$. $\square$

Note that Theorem 3.1 is based on the assumption $ab \leq 1$. If $ab > 1$ we can not find dual frames $\{E_{mb} T_{na} r_1\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb} T_{na} r_2\}_{m,n \in \mathbb{Z}}$ for $L^2(\mathbb{R})$. But if we choose $N \in \mathbb{N}$ such that $\frac{ab}{N} \leq 1$, we can always find dual frames $\{E_{mb} T_{na/N} r_1\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb} T_{na/N} r_2\}_{m,n \in \mathbb{Z}}$. Repeating the proof of Theorem 3.1 with T and U replaced by the preframe operators for $\{E_{mb} T_{na/N} g_1\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb} T_{na/N} g_2\}_{m,n \in \mathbb{Z}}$ now shows that

$$\{E_{mb} T_{na/N} g_1\}_{m,n \in \mathbb{Z}} \cup \{E_{mb} T_{na/N} \Phi^* r_1\}_{m,n \in \mathbb{Z}}$$

and

$$\{E_{mb}T_{na/N}h_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na/N}r_2\}_{m,n\in\mathbb{Z}}$$

form a pair of dual frames for $L^2(\mathbb{R})$. Thus, the given Gabor systems are again subsystems of dual Gabor frames, obtained by oversampling of the original lattice.

As already mentioned, the extension to a pair of dual frames has the advantage (compared to the extension to a tight frame) that the extraction of a square root of a bounded operator. The next example illustrates this in the context of a Gabor system.

Example 3.2 Let $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}}$ be a Gabor Bessel sequence with bound B. Denote the frame operator by S_1 and let $S_2 := BI - S_1$. With $g_2 := b^{1/2}S_2^{1/2}\chi_{[0,a]}$, Example 4.4 in [11] shows that $\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}g_2\}_{m,n\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$. On the other hand, since $\{E_{mb}T_{na}b^{1/2}\chi_{[0,a]}\}_{m,n\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ we can use Proposition 3.1 with $r_1 = r_2 = b^{1/2}\chi_{[0,a]}$ and $\Phi = I - S_1$ to conclude that

$$\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}b^{1/2}(I - S_1)\chi_{[0,a]}\}_{m,n\in\mathbb{Z}}$$

and

$$\{E_{mb}T_{na}g_1\}_{m,n\in\mathbb{Z}} \cup \{E_{mb}T_{na}b^{1/2}\chi_{[0,a]}\}_{m,n\in\mathbb{Z}}$$

are dual frames for $L^2(\mathbb{R})$. Thus, compared to the tight frame approach the construction avoids the use of the square root of the operator S_2. $\square$

4 The wavelet case

Wavelet systems in $L^2(\mathbb{R})$ have the form $\{2^{j/2}\psi(2^jx - k)\}_{j,k\in\mathbb{Z}}$ for some fixed function $\psi \in L^2(\mathbb{R})$. Using the scaling operator,

$$D : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (Df)(x) := 2^{1/2}f(2x), \ x \in \mathbb{R},$$

the wavelet system can be written on the form $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$.

We already mentioned that the unitary extension principle by Ron and Shen allows certain wavelet systems to be extended to tight frames or dual wavelet frames. Important reformulations were later presented by various groups, see [6] and [4]. All of these constructions are based on a multiresolution analysis setup.

In general, the issue of extending a pair of wavelet Bessel sequences to a pair of dual wavelet frames turns out to be significantly more complicated than the corresponding problem for Gabor systems. To illustrate this, consider two Bessel sequences $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ with preframe operators T, U and let $\Phi = I - UT^*$. Then we can copy our approach in Proposition 2.1 and expand $\Phi^* f$ in a pair of dual frames $\{D^jT_kr_1\}_{j,k\in\mathbb{Z}}$, $\{D^jT_kr_2\}_{j,k\in\mathbb{Z}}$. This yields a pair of dual frames, namely,

$$\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}} \cup \{\Phi^* D^jT_kr_1\}_{j,k\in\mathbb{Z}}$$

and

$$\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}} \cup \{D^jT_kr_2\}_{j,k\in\mathbb{Z}}.$$

Here, the problem is that in general $\{\Phi^* D^jT_kr_1\}_{j,k\in\mathbb{Z}}$ does not have wavelet structure. In the following results we present conditions that ensure the wavelet structure. A concrete sufficient condition is presented in Corollary 4.4.

Lemma 4.1 *Let $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$ with preframe operators T, U. Assume there exists a function $\varphi \in L^2(\mathbb{R})$ such that the following two conditions hold:*

(i) $\{D^jT_k\varphi\}_{j,k\in\mathbb{Z}}$ is a wavelet frame that has a dual $\{D^jT_k\widetilde{\varphi}\}_{j,k\in\mathbb{Z}}$;

(ii) $TU^*T_k\varphi = T_kTU^*\varphi$.

Then there exist wavelet systems $\{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$ such that

$$\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}} \text{ and } \{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}} \cup \{D^jT_k\widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$$

form dual frames for $L^2(\mathbb{R})$.

Proof. Consider the operator $\Phi := I - UT^*$. Taking the functions $\varphi, \widetilde{\varphi}$ such that $\{D^jT_k\varphi\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\varphi}\}_{j,k\in\mathbb{Z}}$ are dual wavelet frames, any $f \in L^2(\mathbb{R})$

can be decomposed as

$$\begin{aligned} f &= UT^*f + \Phi f \\ &= \sum_{j,k\in\mathbb{Z}} \langle f, D^jT_k\psi_1\rangle D^jT_k\widetilde{\psi}_1 + \sum_{j,k\in\mathbb{Z}} \langle \Phi f, D^jT_k\varphi\rangle D^jT_k\widetilde{\varphi} \\ &= \sum_{j,k\in\mathbb{Z}} \langle f, D^jT_k\psi_1\rangle D^jT_k\widetilde{\psi_1} + \sum_{j,k\in\mathbb{Z}} \langle f, \Phi^* D^jT_k\varphi\rangle D^jT_k\widetilde{\varphi}. \end{aligned}$$

Put $\psi_2 := \Phi^*\varphi$ and $\widetilde{\psi_2} := \widetilde{\varphi}$. We need to show that the system $\{\Phi^* D^jT_k\varphi\}_{j,k\in\mathbb{Z}}$ has wavelet structure. A standard manipulation shows that the operator D^j commutes with TU^*, so

$$TU^*D^jT_k\varphi = D^jTU^*T_k\varphi. \tag{4.1}$$

Therefore, by the condition (ii),

$$\Phi^*D^jT_k\varphi = (I - TU^*)D^jT_k\varphi = D^j(I - TU^*)T_k\varphi = D^jT_k\Phi^*\varphi,$$

as claimed.

□

A closer look at the conditions in Lemma 4.1 leads to a more concrete condition on the function φ. For $\Phi \subset L^2(\mathbb{R})$, let $\mathcal{S}(\Phi)$ denote the shift-invariant space of Φ defined by

$$\mathcal{S}(\Phi) := \overline{\operatorname{span}}\{T_k\varphi : \varphi \in \Phi, k \in \mathbb{Z}\}.$$

Then

$$\mathcal{S}(\{D^jT_k\psi : j < 0, k \in \mathbb{Z}\}) = \overline{\operatorname{span}}\{T_lD^jT_k\psi : j < 0, k, l \in \mathbb{Z}\}.$$

Theorem 4.2 *Let $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$. Assume there exists a function $\varphi \in L^2(\mathbb{R})$ such that the following two conditions hold:*

(i) *$\{D^jT_k\varphi\}_{j,k\in\mathbb{Z}}$ is a wavelet frame that has a dual $\{D^jT_k\widetilde{\varphi}\}_{j,k\in\mathbb{Z}}$;*

(ii) *$\varphi \in L^2(\mathbb{R}) \ominus \mathcal{S}(\{D^jT_k\widetilde{\psi_1} : j < 0, k \in \mathbb{Z}\})$.*

Then there exist wavelet systems $\{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}}$ *and* $\{D^jT_k\widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$ *such that*

$$\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}\cup\{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}} \text{ and } \{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}\cup\{D^jT_k\widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$$

form dual frames for $L^2(\mathbb{R})$.

Proof. Let T and U denote the preframe operators for $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$, respectively. From Lemma 4.1, it suffices to show that

$$TU^*T_k\varphi = T_kTU^*\varphi. \tag{4.2}$$

The condition (ii) implies that

$$\langle\varphi, T_{-k}D^{j'}T_{k'}\widetilde{\psi_1}\rangle = 0,\ j'<0,\ k,k'\in\mathbb{Z}.$$

Then

$$\begin{aligned}
TU^*T_k\varphi &= \sum_{j',k'\in\mathbb{Z}}\langle T_k\varphi, D^{j'}T_{k'}\widetilde{\psi_1}\rangle D^{j'}T_{k'}\psi_1\\
&= \sum_{j',k'\in\mathbb{Z}}\langle \varphi, T_{-k}D^{j'}T_{k'}\widetilde{\psi_1}\rangle D^{j'}T_{k'}\psi_1\\
&= \sum_{j'\ge 0,k'\in\mathbb{Z}}\langle \varphi, T_{-k}D^{j'}T_{k'}\widetilde{\psi_1}\rangle D^{j'}T_{k'}\psi_1\\
&= T_kT_{-k}\sum_{j'\ge 0,k'\in\mathbb{Z}}\langle \varphi, T_{-k}D^{j'}T_{k'}\widetilde{\psi_1}\rangle D^{j'}T_{k'}\psi_1\\
&= T_k\sum_{j'\ge 0,k'\in\mathbb{Z}}\langle \varphi, T_{-k}D^{j'}T_{k'}\widetilde{\psi_1}\rangle T_{-k}D^{j'}T_{k'}\psi_1.
\end{aligned}$$

Using that $T_{-k}D^{j'}T_{k'} = D^{j'}T_{-2^{j'}k+k'}$, we arrive at

$$\begin{aligned}
TU^*T_k\varphi &= T_k\sum_{j'\ge 0,k'\in\mathbb{Z}}\langle \varphi, D^{j'}T_{-2^{j'}k+k'}\widetilde{\psi_1}\rangle D^{j'}T_{-2^{j'}k+k'}\psi_1\\
&= T_k\sum_{j'\ge 0,k'\in\mathbb{Z}}\langle \varphi, D^{j'}T_{k'}\widetilde{\psi_1}\rangle D^{j'}T_{k'}\psi_1\\
&= T_k\sum_{j',k'\in\mathbb{Z}}\langle \varphi, D^{j'}T_{k'}\widetilde{\psi_1}\rangle D^{j'}T_{k'}\psi_1\\
&= T_kTU^*\varphi.
\end{aligned}$$

Hence (4.2) holds. $\square$

Note that the conditions in Theorem 4.2 are just sufficient conditions for the extension to dual wavelet frames. They are not necessary, as demonstrated by the following example.

Example 4.3 As shown in [1] there exists dual wavelet frames $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ for which

$$\mathcal{S}(\{D^jT_k\widetilde{\psi_1} : j < 0, k \in \mathbb{Z}\}) = L^2(\mathbb{R}).$$

In this case the extension problem has a trivial answer (take $\psi_2 = \widetilde{\psi_2} = 0$), but the conditions (i) and (ii) in Theorem 4.2 can not be satisfied. $\square$

We will now show that the conditions in Theorem 4.2 can be satisfied if the support of the Fourier transform of ψ_1 is sufficiently small. Our convention for the Fourier transform of a function $\psi \in L^1(\mathbb{R})$ is

$$\widehat{\psi}(\gamma) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\gamma}\, dx,$$

with the usual extension to functions in $L^2(\mathbb{R})$.

Corollary 4.4 *Let $\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ be Bessel sequences in $L^2(\mathbb{R})$. Assume that the Fourier transform of $\widetilde{\psi_1}$ satisfies*

$$\textit{supp}\ \widehat{\widetilde{\psi_1}} \subseteq [-1, 1].$$

Then there exist wavelet systems $\{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi_2}\}_{j,k\in\mathbb{Z}}$ such that

$$\{D^jT_k\psi_1\}_{j,k\in\mathbb{Z}} \cup \{D^jT_k\psi_2\}_{j,k\in\mathbb{Z}} \text{ and } \{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}} \cup \{D^jT_k\widetilde{\psi_2}\}_{j,k\in\mathbb{Z}} \quad (4.3)$$

form dual frames for $L^2(\mathbb{R})$.

Proof. Let φ denote the Shannon wavelet, that is,

$$\widehat{\varphi}(\gamma) := \hat{\varphi}_{SH}(\gamma) := \chi_{[-1,-1/2]\bigcup[1/2,1]}(\gamma).$$

Notice that $\{D^jT_k\varphi\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. The condition (ii) implies that $\{D^jT_k\widetilde{\psi_1}\}_{j,k\in\mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$. Since

$$(T_lD^jT_k\widetilde{\psi_1})^\wedge(\gamma) = 2^{-j/2}\widehat{\widetilde{\psi_1}}(\gamma/2^j)e^{-i2\pi\gamma(l+k/2^j)},$$

we see that for each $j < 0, k, l \in \mathbb{Z}$,

$$\text{supp } (T_lD^jT_k\widetilde{\psi_1})^\wedge(\cdot) = \text{supp } \widehat{\widetilde{\psi_1}}(\cdot/2^j) \subset [-2^j, 2^j] \subseteq [-1/2, 1/2].$$

Hence φ is orthogonal to $\mathcal{S}(\{D^jT_k\widetilde{\psi_1} : j < 0, k \in \mathbb{Z}\})$, that is,

$$\varphi \in L^2(\mathbb{R}) \ominus \mathcal{S}(\{D^jT_k\widetilde{\psi_1} : j < 0, k \in \mathbb{Z}\}). \tag{4.4}$$

By Theorem 4.2, the result follows. $\square$

Remark 4.5 A sufficient condition for $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ to be a Bessel sequence is that $\widehat{\psi}$ is bounded and satisfies

$$\begin{aligned} \widehat{\psi}(\gamma) &= O(|\gamma|^\delta) \text{ as } \gamma \to 0, \\ \widehat{\psi}(\gamma) &= O(|\gamma|^{-\frac{1}{2}-\delta}) \text{ as } |\gamma| \to \infty, \end{aligned} \tag{4.5}$$

for some $\delta > 0$, see [10, Theorem 13.0.1]. It is also known that the condition (4.5) is almost necessary: in fact, if $\widehat{\psi}$ is continuous at zero, ψ can only generate a Bessel sequence if $\widehat{\psi}(0) = 0$.

It is natural to ask whether we obtain a result corresponding to the Gabor result in Theorem 3.1(ii) for the wavelet systems in Corollary 4.4. That is, if we also assume that $\widehat{\psi_1}$ is compactly supported, can we find functions ψ_2 and $\widetilde{\psi_2}$ with compactly supported Fourier transforms such that the systems in (4.3) are dual frames? Interestingly, the answer is positive if we strengthen the condition (4.5) slightly:

Corollary 4.6 *In the setup of Corollary 4.4, assume that $\widehat{\psi_1}$ is compactly supported and that*

$$\textit{supp } \widehat{\widetilde{\psi_1}} \subseteq [-1, 1] \setminus [-\epsilon, \epsilon]$$

for some $\epsilon > 0$. Then the functions ψ_2 and $\widetilde{\psi_2}$ can be chosen to have compactly supported Fourier transforms as well.

Proof. Take $\varphi = \widetilde{\varphi} = \varphi_{SH}$ in the proof of Lemma 4.1. Then we just need to show that the function $\widehat{\psi}_2 = \widehat{\Phi^* \varphi}$ is compactly supported. By (4.4), we have

$$\Phi^* \varphi = \varphi - \sum_{j,k \in \mathbb{Z}} \langle \varphi, D^j T_k \widetilde{\psi_1} \rangle D^j T_k \psi_1 = \varphi - \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} \langle \varphi, D^j T_k \widetilde{\psi_1} \rangle D^j T_k \psi_1$$

Now,

$$\langle \varphi, D^j T_k \widetilde{\psi_1} \rangle = \langle \widehat{\varphi}, \mathcal{F} D^j T_k \widetilde{\psi_1} \rangle = \langle \widehat{\varphi}, D^{-j} E_{-k} \widehat{\widetilde{\psi_1}} \rangle.$$

Take $N \in \mathbb{N}$ such that $2^j \epsilon > 1$ for $j > N$. Then $D^{-j} E_{-k} \widehat{\widetilde{\psi_1}}$ is supported outside $[-1, 1]$ for $j > N$, and therefore the above calculation shows that

$$\langle \varphi, D^j T_k \widetilde{\psi_1} \rangle = 0, \ \forall k \in \mathbb{Z}, \text{ if } j > N.$$

Thus,

$$\Phi^* \varphi = \varphi - \sum_{j=0}^{N} \sum_{k \in \mathbb{Z}} \langle \varphi, D^j T_k \widetilde{\psi_1} \rangle D^j T_k \psi_1$$

Taking the Fourier transform yields

$$\widehat{\Phi^* \varphi} = \widehat{\varphi} - \sum_{j=0}^{N} \sum_{k \in \mathbb{Z}} \langle \varphi, D^j T_k \widetilde{\psi_1} \rangle D^{-j} E_{-k} \widehat{\widetilde{\psi_1}},$$

which is clearly compactly supported. □

Note that the results presented here are complementary to the traditional extension principles in wavelet analysis. In fact, the classical applications of the unitary extension principle and its variants deal with wavelet systems generated by compactly supported functions. In contrast, the concrete manifestations of our results in Corollary 4.4 deals with extension of wavelet systems generated by functions whose Fourier transform has compact support.

5 Appendix A

For the case $ab = 1$ it is well known that there does not exist a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with a compactly supported continuous generator g. However, there exist tight frames generated by a compactly supported function. For the case $ab < 1$ we can find Gabor frames with compactly supported generators of arbitrary smoothness, with dual generators enjoying the same properties:

Example 5.1 Assume that $ab < 1$. Take $\epsilon \in [0, 1/2]$ such that $a + 2\epsilon < 1/b$, and choose a function $g \in L^2(\mathbb{R})$ such that

- $\operatorname{supp} g \subseteq [0, a + 2\epsilon]$;
- $g = 1$ on $[\epsilon, a + \epsilon]$;
- $g \in C^\infty(\mathbb{R})$;
- $||g||_\infty = 1$.

Then the function

$$G(x) := \sum_{n\in\mathbb{Z}} |g(x - na)|^2$$

is bounded below by 1 and bounded above by 2. It is well known (see, e.g., Cor. 9.1.7 in [3]) that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame with bounds $1/b, 2/b$. Letting S denote the frame operator, the canonical dual frame is given by $\{E_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}}$, where

$$S^{-1}g = \frac{b}{G}\, g.$$

By construction, $S^{-1}g$ is compactly supported and belongs to $C^\infty(\mathbb{R})$. $\square$

Note that this construction proves the existence of the functions r_1, r_2 in the proof of Theorem 3.1. Thus, if $ab < 1$, the function h_2 can always be chosen to be compactly supported and C^∞.

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