Regularity of dual Gabor windows

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Abstract

We present a construction of dual windows associated with Gabor frames with compactly supported windows. The size of the support of the dual windows are comparable to that of the given window. Under certain conditions we prove that there exist dual windows with higher regularity than the canonical dual window. On the other hand, there are cases where no differentiable dual window exists, even in the overcomplete case. As a special case of our results we show that there exists a common smooth dual window for an interesting class of Gabor frames. In particular, for any value of $K \in \mathbb{N}$ there is a smooth function h which simultaneously is a dual window for all B-spline generated Gabor frames $\{E_{mb}T_nB_N(\frac{x}{2})\}_{m,n\in\mathbb{N}}$ for B-splines B_N of order $N = 1, \dots, 2K+1$ with a fixed and sufficiently small value of b.

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1 Introduction

A frame $\{f_k\}$ in a separable Hilbert space $\mathcal{H}$ leads to expansions of arbitrary elements $f \in \mathcal{H}$, in a similar fashion as the well known orthonormal bases. More precisely, there exists at least one so-called *dual frame,* i.e., a frame $\{h_k\}$ such that

$$f = \sum \langle f, h_k \rangle f_k, \ \forall f \in \mathcal{H}.$$

Unless $\{f_k\}$ is a basis, the dual $\{h_k\}$ is not unique. This makes it natural to search for duals with special prescribed properties. In this paper we will consider Gabor frames with

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translation parameter $a = 1$, i.e., frames for $L^2(\mathbb{R})$ that for a certain fixed parameter $b > 0$ and a fixed function $g \in L^2(\mathbb{R})$ have the form $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}} := \{e^{2\pi imbx}g(x-n)\}_{m,n\in\mathbb{Z}}$. The function g is called the *window function.* We will construct dual frames of the form $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx}h(x-n)\}_{m,n\in\mathbb{Z}}$ for a suitable function $h \in L^2(\mathbb{R})$, to be called the *dual window.*

It is known that a frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ will be *overcomplete* if $b < 1$, with the redundancy increasing when b decreases. Thus, the dual window is not unique. We will investigate whether this freedom can be used to find dual windows with higher regularity than the canonical dual and comparable size of the support. We will present cases where this is possible, and other cases where it is not. As a special case of our results we show that there are certain classes of interesting frames that have the *same* dual window. For example, for any value of $K \in \mathbb{N}$ there is a smooth function h which simultaneously is a dual window for all B-spline generated Gabor frames $\{E_{mb}T_n B_N(\frac{x}{2})\}_{m,n\in\mathbb{N}}$ for B-splines B_N of order $N = 1, \dots, 2K+1$ with a fixed and sufficiently small value of b.

Just to give the reader an impression of the results to come, consider the B-spline B_2. The function B_2 is continuous but not differentiable at the points $0, \pm 1$. The Gabor system $\{E_{mb}T_n B_2\}_{m,n\in\mathbb{Z}}$ is a frame for all sufficiently small values of $b > 0$. As b tends to zero, the redundancy of the frame $\{E_{mb}T_n B_2\}_{m,n\in\mathbb{Z}}$ increases, meaning that we get more and more freedom in the choice of a dual window h. However, we will show that it is impossible to find a differentiable dual window h supported on $\operatorname{supp} B_2 = [-1, 1]$, regardless of the considered $b > 0$. On the other hand, by a seemingly innocent scaling we obtain the function $g(x) := B_2(x/2)$. Again, $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a frame for all sufficiently small values of $b > 0$. But in contrast with the situation for B_2, we will show that one can find infinitely often differentiable dual windows h that are supported on $\operatorname{supp} g = [-2, 2]$, for any value of $b \in]0, 1/2]$. These examples illustrate that the question of differentiability of the dual windows is a nontrivial issue. The examples will be derived as a consequence of more general results, see Example 4.4.

The above results will be based on a construction of dual windows, to be presented in Section 2. In Section 3 we consider a particular case where it is possible to obtain smooth dual windows, regardless of the regularity of the given window g. This is much more than one can hope for in the general case. A general approach to the question of differentiability of the dual windows is given in Section 5. Since the necessary conditions are quite involved and not very intuitive, we first, in Section 4, state a version for the case of windows g that are supported in $[-1, 1]$. For this case we can provide concrete examples demonstrating that the desired conclusions might fail if any of the constraints is removed.

Note that a complementary approach to duality for Gabor frames that also deals with the issue of regularity is considered by R. Laugesen [8] and I. Kim [6]. For more information about the theory for Gabor analysis and its applications, see [3, 4, 5].

2 Construction of dual frames

In the literature, various characterizations of the pairs of dual Gabor frames are available. For general frames, Li gave a characterization in [9], which in the special case of Gabor frames also lead to a class of dual Gabor frames; later, in [2], it was shown that these duals actually characterize all duals. In order to start our analysis, we need the duality conditions for Gabor frames by Ron and Shen [11], resp. Janssen [7]:

Lemma 2.1 *Two Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x+n/b+ka)}h(x+ka) \;=\; b\delta_{n,0},\ a.e.\ x\in[0,a]. \tag{2.1}$$

We will use the following to apply Lemma 2.1:

Lemma 2.2 *Let $\tilde{G}$ be a real-valued bounded function, and assume that for some constant $A>0$,*

$$|\tilde{G}(x-1)|+|\tilde{G}(x)|\geq A,\ x\in[0,1]. \tag{2.2}$$

Then there exists a real-valued bounded function $\tilde{H}$ with supp $\tilde{H}\subseteq$ supp $\tilde{G}\bigcap[-1,1]$ such that

$$\tilde{G}(x-1)\tilde{H}(x-1)+\tilde{G}(x)\tilde{H}(x)=1,\ x\in[0,1]. \tag{2.3}$$

Proof. Consider $x\in[0,1]$. We will define $\tilde{H}(x)$ and $\tilde{H}(x-1)$ simultaneously. In case $|\tilde{G}(x-1)|\geq A/2$, put $\tilde{H}(x)=0$ and $\tilde{H}(x-1)=\frac{1}{\tilde{G}(x-1)}$. On the other hand, if $|\tilde{G}(x-1)|<A/2$, we know that $|\tilde{G}(x)|\geq A/2$. In this case, put $\tilde{H}(x-1)=0$ and $\tilde{H}(x)=\frac{1}{\tilde{G}(x)}$. Clearly, we can take $\tilde{H}=0$ outside $[-1,1]$. □

We will now present a general result about the existence of frames with a dual window of a special form. Note that, in contrast with most results from the literature, we do not need that the integer-translates of the window function form a partition of unity.

Associated to a function g with support on an interval $[-(2K+1),2K+1],\ K\in\mathbb{N}$, we will in the rest of the paper use the function

$$\tilde{G}(x):=\sum_{k\in\mathbb{Z}} g(x+2k),\ x\in\mathbb{R}. \tag{2.4}$$

Due to the periodicity of $\tilde{G}$, we are mainly interested in $x\in[-1,1]$. Note that by the compact support of g,

$$\tilde{G}(x)=\sum_{k=-K}^{K} g(x+2k),\ x\in[-1,1]. \tag{2.5}$$

Theorem 2.3 *Let $K\in\mathbb{N}\bigcup\{0\}$ and let $b\in]0,\frac{1}{4K+2}]$. Let g be a real-valued bounded function with supp $g\subseteq[-(2K+1),2K+1]$, for which*

$$\left|\sum_{n\in\mathbb{Z}} g(x+n)\right|\geq A,\ x\in[0,1], \tag{2.6}$$

for a constant $A>0$. Then the following hold:

(i) The function $\tilde{G}$ in (2.4) satisfies the conditions in Lemma 2.2.

(ii) Take $\tilde{H}$ as in Lemma 2.2, and let

$$h(x) = b \sum_{k=-K}^{K} T_{2k}\tilde{H}(x). \tag{2.7}$$

Then $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_nh\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$, and h is supported in $[-(2K+1), 2K+1]$.

Proof. To prove (i), we note that by (2.6) and the definition of $\tilde{G}$, for $x \in [0,1]$,

$$\begin{aligned} \left|\tilde{G}(x-1)\right| + \left|\tilde{G}(x)\right| &= \left|\sum_{k\in\mathbb{Z}} g(x-1+2k)\right| + \left|\sum_{k\in\mathbb{Z}} g(x+2k)\right| \\ &\geq \left|\sum_{k\in\mathbb{Z}} g(x+k)\right| \geq A. \end{aligned}$$

Thus $\tilde{G}$ satisfies the condition (2.2) in Lemma 2.2. Also, it is clear that $\tilde{G}$ is bounded, real-valued, and 2-periodic. Therefore we can choose a function $\tilde{H}$ which is supported in $[-1,1]$ and such that

$$\tilde{G}(x-1)\tilde{H}(x-1) + \tilde{G}(x)\tilde{H}(x) = 1, \; x \in [0,1]. \tag{2.8}$$

Define h as in (2.7). In order to prove (ii) we will apply Lemma 2.1. By assumption, the function g has compact support and is bounded; by the construction, the function h shares these properties. It follows that $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_nh\}_{m,n\in\mathbb{Z}}$ are Bessel sequences. In order to verify that these sequences form dual frames, we need to check that for $x \in [0,1]$,

$$\sum_{k\in\mathbb{Z}} \overline{g(x+n/b+k)}h(x+k) \;=\; b\delta_{n,0}, \; a.e. \; x \in [0,1]. \tag{2.9}$$

By assumption and construction, g and h have support in $[-(2K+1), 2K+1]$; thus (2.9) is satisfied for $n \neq 0$ whenever $\frac{1}{b} \geq 4K+2$, *i.e.*, if $b \in \left]0, \frac{1}{4K+2}\right]$. For $n=0$, and using the compact support of g, we need to check that

$$\sum_{k=-2K-1}^{2K} g(x+k)h(x+k) = b, \; x \in [0,1]. \tag{2.10}$$

For each $k \in \{-K, -K+1, \cdots, K\}$, if $x \in [0,1]$, then

$$x-1+2k \in [2k-1, 2k], \; x+2k \in [2k, 2k+1];$$

thus we have

$$h(x-1+2k) = b\tilde{H}(x-1), \; h(x+2k) = b\tilde{H}(x).$$

This together with (2.5) and (2.8) implies, for $x \in [0,1]$,

$$\begin{aligned}\sum_{k=-2K-1}^{2K} g(x+k)h(x+k) &= \sum_{k=-K}^{K} g(x-1+2k)h(x-1+2k) + \sum_{k=-K}^{K} g(x+2k)h(x+2k) \\ &= b\sum_{k=-K}^{K} g(x-1+2k)\tilde{H}(x-1) + b\sum_{k=-K}^{K} g(x+2k)\tilde{H}(x) \\ &= b\left(\tilde{G}(x-1)\tilde{H}(x-1)+\tilde{G}(x)\tilde{H}(x)\right) = b.\end{aligned}$$

Hence (2.10) holds. Therefore $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$. □

3 Smooth dual windows for a class of Gabor frames

Before we start the general analysis of the dual windows in Theorem 2.3 we will consider a particular case, where we can construct smooth compactly supported dual windows, regardless of the regularity of the window itself.

Theorem 3.1 *Let $K \in \mathbb{N}\bigcup\{0\}$ and let $b \in]0, \frac{1}{4K+2}]$. Let g be a real-valued bounded function with supp $g \subseteq [-(2K+1), 2K+1]$, for which*

$$\sum_{n\in\mathbb{Z}} g(x+2n) = 1, \ x \in [-1,1]. \tag{3.1}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be any bounded function for which

$$f(x) = 0, \ x \le 0, \quad f(x) = 1, \ x \ge 1. \tag{3.2}$$

Define the function $\tilde{H}$ by

$$\tilde{H}(x) = \begin{cases} \frac{1}{2}f(2(x+1)), & -1 \le x < -1/2, \\ 1-\frac{1}{2}f(-2x), & -1/2 \le x < 0, \\ 1-\frac{1}{2}f(2x), & 0 \le x < 1/2, \\ \frac{1}{2}f(2(1-x)), & 1/2 \le x < 1, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.3}$$

and let

$$h(x) := b\sum_{k=-K}^{K} T_{2k}\tilde{H}(x). \tag{3.4}$$

Then the following holds:

(i) h is a symmetric function with supp $h \subseteq [-(2K+1), 2K+1]$.

(ii) $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_nh\}_{m,n\in\mathbb{Z}}$ *are dual frames for* $L^2(\mathbb{R})$.

(iii) If f *is chosen to be smooth, then the function* $\widetilde{H}$ *in* (3.3) *is smooth, and consequently the dual window* h *in* (3.4) *is smooth as well.*

Proof. By the assumption (3.1), we have

$$\tilde{G}(x) = 1, \ x \in [-1, 1]. \tag{3.5}$$

Take $\tilde{H}$ as in (3.3). For $x \in]0, 1/2[$,

$$\tilde{H}(x-1) + \tilde{H}(x) = f(2x)/2 + (1 - f(2x)/2) = 1;$$

for $x \in]1/2, 1[$,

$$\tilde{H}(x-1) + \tilde{H}(x) = (1 - f(-2(x-1))/2) + f(2(1-x))/2 = 1.$$

That is, $\tilde{H}(x-1) + \tilde{H}(x) = 1, \ x \in [0, 1]$. Combining this with (3.5) yields

$$\tilde{G}(x-1)\tilde{H}(x-1) + \tilde{G}(x)\tilde{H}(x) = \tilde{H}(x-1) + \tilde{H}(x) = 1, \ x \in [0, 1].$$

By Theorem 2.3, h is a dual window of g. By construction, h is a symmetric function with $\text{supp } h \subseteq [-(2K+1), 2K+1]$. The result in (iii) follows by direct investigation of the derivatives at $-1, -1/2, 0, 1/2, 1$. $\square$

An example of a smooth function f satisfying condition (3.2) is (see [10, p.36])

$$f(x) = \begin{cases} \exp\left[-\left\{\exp\left[x/(1-x)\right] - 1\right\}^{-1}\right], & 0 < x < 1, \\ 0, & x \le 0, \\ 1, & x \ge 1. \end{cases} \tag{3.6}$$

As noted in the introduction, the possibility of constructing a smooth dual window is a significant improvement compared to the use of the canonical dual window, which might not even be continuous. We return to this point in Example 4.4.

Another interesting feature of the construction in Theorem 3.1 is that the dual window h in (3.4) is *independent* of the window g that generates the frame $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$! In other words, we can construct a window h that generates a dual frame for all the frames $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ satisfying the conditions in Theorem 3.1 for fixed values of K and b.

Corollary 3.2 *Let* $K \in \mathbb{N}$ *and let* $b \in]0, \frac{1}{4K+2}]$. *Consider a bounded real-valued function* ϕ *that is supported on* $[-K, K]$ *and satisfies the partition of unity condition,*

$$\sum_{n\in\mathbb{Z}} \phi(x-n) = 1, \ x \in \mathbb{R}. \tag{3.7}$$

Then the function $g(x) := \phi(x/2)$ *generates a Gabor frame* $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$. *Choosing* $\tilde{H}$ *as in* (3.3), *the function* h *in* (3.4) *is a dual window of* g.

Proof. By the choice of the function ϕ, supp $g \subseteq [-2K, 2K] \subseteq [-2K-1, 2K+1]$, and

$$\sum_{n\in\mathbb{Z}} g(x-2n) = \sum_{n\in\mathbb{Z}} \phi(x/2-n) = 1.$$

Choose $\tilde{H}$ as in (3.3). By Theorem 3.1, the function h defined by $h(x) = b\sum_{k=-K}^{K} T_{2k}\tilde{H}(x)$ is a dual window of g. □

It is known that the partition of unity condition (3.7) is satisfied for a large class of functions, e.g., any scaling function for a multiresolution analysis. As a concrete example, recall that the centered B-splines B_N, $N \in \mathbb{N}$ are given inductively by $B_1 = \chi_{[-1/2,1/2]}$, $B_{N+1} = B_N * B_1$. Any B-spline satisfies the partition of unity condition, and the B-spline B_N has support on the interval $[-N/2, N/2]$. Thus, for each fixed value of $K \in \mathbb{N}$, the function h in (3.4) is a dual window for each of the B-splines B_N, $N = 1, \ldots, 2K$ and a fixed choice of $b \le \frac{1}{4K+2}$.

4 Regularity of the dual windows if $\operatorname{supp} g \subseteq [-1, 1]$

Based on Theorem 2.3 we now aim at a general analysis of the relationship between the regularity of a window g and the associated dual windows h with comparable support size. We will exhibit cases where the smoothness can be increased, and other cases where these dual windows can not have higher smoothness than the window itself. The general version of our result, to be stated in Theorem 5.1, is quite complicated and the role of the conditions not intuitively clear. Therefore we will first present, in Theorem 4.1, the corresponding version for windows g that are supported in $[-1, 1]$. An advantage of this approach is that for each of the requirements in Theorem 4.1 we can provide an example showing that the desired conclusion might fail if the condition is removed.

Given a function g that is supported in $[-1, 1]$, let

$$Z \;:=\; \{x \in [-1,1] : g(x) = 0\} \tag{4.1}$$

and

$$E \;:=\; \{x \in [-1,1] : g \text{ is not differentiable at } x\} \tag{4.2}$$

Theorem 4.1 *Let g be a real-valued bounded function with* $\operatorname{supp} g \subseteq [-1, 1]$. *Assume that for some constant $A > 0$,*

$$|g(x-1)|^2 + |g(x)|^2 \ge A, \;\; x \in [0, 1]. \tag{4.3}$$

Then the following assertions hold:

(1) *If g is not differentiable at 0, then there does not exist a differentiable dual window h with supp $h \subseteq [-1, 1]$ for any $b > 0$;*

(2) *Assume that g is differentiable at* 0. *Assume further that the set* Z *is a finite union of intervals and points not containing* 0, *that the set* E *is finite, and that*

(a) $E\bigcap(E+1)=\emptyset$;

(b) $Z\bigcap(E\pm 1)=\emptyset$.

Then, for any $b\in \left]0,\frac{1}{2}\right]$, *there exists a differentiable dual window* h *with* $\operatorname{supp} h \subseteq \operatorname{supp} g$.

Note that (4.3) is a necessary condition for $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ being a frame; thus it is not a restriction in this context. To support the conditions in Theorem 4.1 we will now provide a series of examples, where just one of these conditions breaks down and the conclusion in Theorem 4.1 fails. We first give an example where condition (a) is not satisfied.

Example 4.2 Consider the function

$$g(x):=\begin{cases} 2x+2, & x\in[-1,-1/2];\\ 1, & x\in[-1/2,1/2];\\ -2x+2, & x\in[1/2,1];\\ 0, & x\notin[-1,1].\end{cases}$$

Then $K=0,\ Z=\{\pm 1\}$ and $E=\{\pm 1/2,\pm 1\}$, so $E\bigcap(E+1)=\{1/2\},\ Z\bigcap(E\pm 1)=\emptyset$. Hence g satisfies condition (b) but not (a) in Theorem 4.1. Now we will show that g does not have a differentiable dual of any form. Suppose that there exists such a dual h. By the duality condition we obtain

$$g(x-1)h(x-1)+g(x)h(x)=b,\ x\in[0,1]. \tag{4.4}$$

Letting $D_{-}g$ and $D_{+}g$ denote the left, resp. right derivatives of g, we note that

$$g(-1/2)=g(1/2)=1,\ D_{-}g(-1/2)=2, D_{+}g(-1/2)=0,\ D_{-}g(1/2)=0, D_{+}g(1/2)=-2. \tag{4.5}$$

Taking the left and right derivative of (4.4) at $x=1/2$, this implies that

$$Dh(-1/2)+Dh(1/2)=-2h(-1/2) \text{ and } Dh(-1/2)+Dh(1/2)=2h(1/2).$$

But by the conditions (4.4) and (4.5), we have $h(-1/2)+h(1/2)=b$. This is a contradiction. Thus, a differentiable dual window does not exist. □

Note that the conclusion in Example 4.2 is even stronger than what we asked for: no dual window at all can be differentiable, regardless of its form and support size. In the next example condition (b) in Theorem 4.1 is not satisfied and the conclusion breaks down.

Example 4.3 Consider

$$g(x) := \begin{cases} \frac{5}{2}x + \frac{3}{2}, & x \in [-3/5, -1/5]; \\ 1, & x \in [-1/5, 1/5]; \\ -\frac{5}{2}x + \frac{3}{2}, & x \in [1/5, 3/5]; \\ 0, & x \notin [-3/5, 3/5]. \end{cases}$$

Easy considerations as in Example 4.2 show that g satisfies condition (a) but not (b) in Theorem 4.1, and that g does not have a differentiable dual of any form. We leave the details to the interested reader. □

Let us now provide the details for the example mentioned in the introduction:

Example 4.4 Consider the B-spline B_2, which is continuous but not differentiable at the points $0, \pm 1$. It is an easy consequence of the results in the literature (see, e.g., [1]) that the Gabor system $\{E_{mb}T_nB_2\}_{m,n\in\mathbb{Z}}$ is a frame for all sufficiently small values of $b > 0$. As b tends to zero, the redundancy of the frame $\{E_{mb}T_nB_2\}_{m,n\in\mathbb{Z}}$ increases, meaning that we get more and more freedom in the choice of h. However, since B_2 is non-differentiable at $x = 0$, Theorem 4.1 implies that none of the dual windows supported on $[-1, 1]$ are differentiable, for any $b \in]0, 1/2]$.

On the other hand, consider the scaled B-spline $g(x) := B_2(x/2)$. Again, $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ is a frame for all sufficiently small values of $b > 0$. The requirements in Theorem 3.1 are satisfied for $b \in]0, 1/2]$, implying that infinitely often differentiable dual windows exists. □

5 Regularity of the dual windows in the general case

We will now present the general version of Theorem 4.1. The main difference between the results is that the conditions in the general version are stated in terms of the function $\tilde{G}$ in (2.4) rather than g itself. The proof is in Appendix A. Given a compactly supported function $g : \mathbb{R} \to \mathbb{C}$, define $\tilde{G}$ as in (2.4) and let

$$\widetilde{Z} \ := \ \{x \in [-1, 1] : \tilde{G}(x) = 0\}$$

and

$$\widetilde{E} \ := \ \{x \in [-1, 1] : \widetilde{G}\chi_{[-1,1]} \text{ is not differentiable at } x\}$$

Theorem 5.1 *Let $K \in \mathbb{N} \bigcup \{0\}$ and let $b \in]0, \frac{1}{4K+2}]$. Let g be a real-valued bounded function with $\operatorname{supp} g \subseteq [-(2K+1), 2K+1]$. Define $\tilde{G}$ as in (2.4). Then the following assertions hold:*

(1) *If $\tilde{G}$ is not differentiable at 0, then there does not exist a differentiable dual window h defined as in (2.7);*

Assume that for some constant $A > 0$,

$$|\tilde{G}(x-1)| + |\tilde{G}(x)| \geq A, \; x \in [0,1], \tag{5.1}$$

and that the set $\widetilde{Z}$ *is a finite union of intervals and points not containing* 0.

(2) *Assume that* $\tilde{G}$ *is differentiable at* 0, *that the set of points* $\tilde{E}$ *is finite, and that*

(a) $\tilde{E} \bigcap \left(\tilde{E} - 1\right) = \emptyset$;

(b) $\tilde{G}(x) \neq 0, \; x \in \left(\tilde{E} - 1\right) \bigcup \left(\tilde{E} + 1\right)$.

Then $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *and there exists a differentiable dual window* h *of the form* (2.7).

Note that the conditions in Theorem 5.1 (2) are void if $\tilde{G}$ is differentiable, i.e., the standing assumptions alone imply the existence of a differentiable dual window.

For non-negative functions the conditions in Theorem 5.1 can be formulated in an easier way, where we again refer directly to properties of the function g rather than $\widetilde{G}$.

Corollary 5.2 *Let* $K \in \mathbb{N}\bigcup\{0\}$ *and let* $b \in]0, \frac{1}{4K+2}]$. *Let* g *be a non-negative bounded function with* $\operatorname{supp} g \subseteq [-(2K+1), 2K+1]$. *Assume that for some constant* $A > 0$,

$$\sum_{k=-2K-1}^{2K} g(x+k) \geq A, \; x \in [0,1]. \tag{5.2}$$

Assume that the set Z *of zeros of* g *on* $[-(2K+1), 2K+1]$ *is a finite union of intervals and points, that* g *is differentiable except on a finite set* E *of points, and that the sets* E *and* Z *satisfy the following conditions:*

(a) $0 \notin \left(\bigcap_{k=-K}^{K}(Z-2k)\right) \bigcup \left(\bigcup_{k=-K}^{K}(E-2k)\right)$;

(b) $\left(\bigcup_{k=-K}^{K}(E-2k)\right) \bigcap \left(\bigcup_{k=-K}^{K}(E-2k+1)\right) = \emptyset$;

(c) $\left(\bigcap_{k=-K}^{K}(Z-2k)\right) \bigcap \left(\bigcup_{k=-K}^{K}(E-2k\pm 1)\right) = \emptyset$.

Then $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *and there exists a differentiable dual window* h *supported on* $[-(2K+1), 2K+1]$.

Proof. We check condition (2) in Theorem 5.1. Let

$$\tilde{G}(x) := \sum_{k\in\mathbb{Z}} g(x+2k).$$

By an argument similar to the one at the beginning of the proof of Theorem 2.3, (5.2) together with (2.5) implies

$$|\tilde{G}(x-1)| + |\tilde{G}(x)| \geq A, \ x \in [0,1].$$

Since g is non-negative, the zeros of $\tilde{G}$ restricted to $[-1,1]$ is

$$\tilde{Z} := \left(\bigcap_{k=-K}^{K} (Z-2k) \right) \bigcap [-1,1].$$

A direct calculation shows that if g is differentiable except on the set E, then $\tilde{G}$ is differentiable outside the set $\bigcup_{k=-K}^{K}(E-2k)$. Let

$$\tilde{E} := \left(\bigcup_{k=-K}^{K} (E-2k) \right) \bigcap [-1,1].$$

Then the sets $\tilde{E}$ and $\tilde{Z}$ satisfy the conditions (a) and (b) of (2) in Theorem 5.1. Hence there exists a differentiable dual window h. □

Example 5.3 Let

$$g(x) := \left(\frac{5}{2}x+2\right)\chi_{[-\frac{4}{5},-\frac{2}{5}]}(x) + \chi_{[-\frac{2}{5},\frac{2}{5}]}(x) + \left(-\frac{5}{2}x+2\right)\chi_{[\frac{2}{5},\frac{4}{5}]}(x).$$

A direct calculation shows that

$$E = \left\{\pm\frac{2}{5}, \pm\frac{4}{5}\right\}, \ E+1 = \left\{\frac{1}{5}, \frac{3}{5}, \frac{7}{5}, \frac{9}{5}\right\}, \ E-1 = \left\{-\frac{9}{5}, -\frac{7}{5}, -\frac{3}{5}, -\frac{1}{5}\right\}$$

and

$$Z = \left[-1, -\frac{4}{5}\right] \bigcup \left[\frac{4}{5}, 1\right].$$

Thus the conditions (a)-(c) in Corollary 5.2 are satisfied. Hence, for any $b \in]0, \frac{1}{2}]$, there exists a differentiable dual window h.

Appendix: proof of Theorem 5.1:

The full proof of Theorem 5.1 is notationally complicated. We will therefore formulate the proof for a function g for which

- $K = 0$ in (2.5), *i.e.*, $\tilde{G}(x) = g(x)$, $x \in [-1,1]$;
- $\tilde{G}$ is differentiable except at one point, *i.e.*, $\tilde{E} = \{c\}$;

- The zeroset of $\widetilde{G}$ within $]-1,1[$ consists of just one interval $[a,b]$ (containing the degenerate case of just one point $a=b$ as a special case), *i.e.*, $\tilde{Z}=\{-1\}\bigcup[a,b]\bigcup\{1\}$.

We leave the obvious modifications to the general case to the reader.

We use the following abbreviation:

$$\psi(a^-):=\lim_{x\to a^-}\psi(x),\quad \psi(a^+):=\lim_{x\to a^+}\psi(x).$$

Proof of Theorem 5.1: (1) We will prove the contra-positive result, so suppose that a dual window h defined as in (2.7) with $K=0$, *i.e.*,

$$h(x)=b\tilde{H}(x)$$

with $\mathrm{supp}\tilde{H}\subseteq[-1,1]$, is differentiable on $\mathbb{R}$. Then $\tilde{H}$ is also differentiable on $\mathbb{R}$. Then we have

$$\tilde{H}(-1)=\tilde{H}(1)=D\tilde{H}(-1)=D\tilde{H}(1)=0. \tag{5.3}$$

The duality condition can for $n=0$ be written as

$$g(x-1)h(x-1)+g(x)h(x)=b,\ x\in[0,1], \tag{5.4}$$

or, in terms of $\tilde{G}$ and $\tilde{H}$,

$$\tilde{G}(x-1)\tilde{H}(x-1)+\tilde{G}(x)\tilde{H}(x)=1,\ x\in[0,1]. \tag{5.5}$$

It follows from (5.3) that

$$\tilde{G}(0)\tilde{H}(0)=1.$$

Putting this into (5.5), we have

$$\tilde{G}(x-1)\ \tilde{H}(x-1)+\tilde{G}(x)\left(\tilde{H}(x)-\tilde{H}(0)\right)+\tilde{H}(0)\left(\tilde{G}(x)-\tilde{G}(0)\right)=0,\ x\in]0,1[,$$

or, by dividing with x,

$$\tilde{G}(x-1)\frac{\tilde{H}(x-1)}{x}+\tilde{G}(x)\left(\frac{\tilde{H}(x)-\tilde{H}(0)}{x}\right)+\tilde{H}(0)\left(\frac{\tilde{G}(x)-\tilde{G}(0)}{x}\right)=0,\ x\in]0,1[.$$

This implies that

$$\frac{\tilde{G}(x)-\tilde{G}(0)}{x}=-\frac{\tilde{G}(x-1)\frac{\tilde{H}(x-1)}{x}+\tilde{G}(x)\left(\frac{\tilde{H}(x)-\tilde{H}(0)}{x}\right)}{\tilde{H}(0)},\ x\in]0,1[.$$

Using (5.3) it now follows that

$$D_+\tilde{G}(0)=-\frac{\tilde{G}(0^+)D_+\tilde{H}(0)+\tilde{G}((-1)^+)D_+\tilde{H}(-1)}{\tilde{H}(0)}=-\frac{\tilde{G}(0^+)D\tilde{H}(0)}{\tilde{H}(0)}.$$

Similarly,

$$D_-\tilde{G}(0) = -\frac{\tilde{G}(0^-)D\tilde{H}(0)}{\tilde{H}(0)}.$$

The conditions (5.3), (5.4) and the continuity of $\tilde{H}$ show that $\tilde{G}$ is continuous at $x = 0$. Hence $D_+\tilde{G}(0) = D_-\tilde{G}(0)$, which implies that $\tilde{G}$ is differentiable at 0.

(2): We construct a real-valued differentiable function $\tilde{H}$ with supp$\tilde{H} \subseteq [-1, 1]$ so that

$$\tilde{G}(x-1)\tilde{H}(x-1) + \tilde{G}(x)\tilde{H}(x) = 1, x \in [0, 1].$$

Let

$$\tilde{Z} := \{-1\} \bigcup [a, b] \bigcup \{1\} \text{ and } \tilde{E} := \{c\}.$$

There are various scenarios concerning the location of the points a, b, c. They are treated in a similar way, and we will assume that $-1 < a \le b < 0 < c < 1$. By the condition (2) in Theorem 5.1, one of the following cases occurs:

$$(1)\ b < c - 1 < 0; \quad (2)\ -1 < c - 1 < a.$$

The cases are similar, so we only consider the case (1), *i.e.*,

$$-1 < a \le b < c - 1 < 0 < c < 1. \tag{5.6}$$

Let

$$\tilde{W} := \tilde{Z} \bigcup \tilde{E} = \{-1\} \bigcup [a, b] \bigcup \{c\} \bigcup \{1\}.$$

and

$$\tilde{V} := \left(\tilde{W} \bigcap [-1, 0]\right) \bigcup \left((\tilde{W} - 1) \bigcap [-1, 0]\right) = \left(\{-1\} \bigcup [a, b]\right) \bigcup \left(\{c - 1\} \bigcup \{0\}\right)$$

We define $\tilde{H}(x)$ on $[-1, 1]$ as follows: first, we define $h(x)$ on $\tilde{V} \bigcup \left(\tilde{V} + 1\right)$ by

$$\tilde{H}(x) := \begin{cases} 0, & x \in \tilde{W} \bigcap [-1, 0] = \{-1\} \bigcup [a, b]; \\ \frac{1}{\tilde{G}(x)}, & x \in (\tilde{W} - 1) \bigcap [-1, 0] = \{c - 1\} \bigcup \{0\}; \\ \frac{1}{\tilde{G}(x)}, & x \in (\tilde{W} + 1) \bigcap [0, 1] = \{0\} \bigcup [a + 1, b + 1]; \\ 0, & x \in \tilde{W} \bigcap [0, 1] = \{c\} \bigcup \{1\}, \end{cases} \tag{5.7}$$

which is well defined by (5.6). Then

$$\tilde{G}(x-1)\tilde{H}(x-1) + \tilde{G}(x)\tilde{H}(x) = 1,\ x \in \tilde{V} + 1; \tag{5.8}$$

We now extend $\tilde{H}(x)$ on $[-1, 1] \backslash \left(\tilde{V} \bigcup \left(\tilde{V} + 1\right)\right) = \left([-1, 0] \setminus \tilde{V}\right) \bigcup \left([0, 1] \setminus \left(\tilde{V} + 1\right)\right)$: choose $\tilde{H}(x)$ on $[-1, 0] \setminus \tilde{V}$ so that $\tilde{H}$ is differentiable on $[-1, 0] \setminus \tilde{V}$ and

$$D_+\tilde{H}(-1) = D_-\tilde{H}(a) = D_+\tilde{H}(b) = 0 \tag{5.9}$$

$$D\tilde{H}(c - 1) = D\left(\frac{1}{\tilde{G}}\right)(c - 1) \tag{5.10}$$

$$D_-\tilde{H}(0) = D\left(\frac{1}{\tilde{G}}\right)(0). \tag{5.11}$$

Then (5.7) and (5.9)-(5.11) imply that $\tilde{H}$ is differentiable on $]-1,0[$ and that

$$D_+\tilde{H}(-1) = 0,\ D_-\tilde{H}(0) = D\left(\frac{1}{\tilde{G}}\right)(0). \tag{5.12}$$

We then define $\tilde{H}(x)$ on $[0,1]\setminus(\tilde{V}+1)$ by

$$\tilde{H}(x) = \frac{1-\tilde{G}(x-1)\tilde{H}(x-1)}{\tilde{G}(x)}, \tag{5.13}$$

which is well defined since

$$\tilde{W}\bigcap[0,1] = \{c,1\} \subseteq (\tilde{V}+1).$$

This implies that

$$\tilde{G}(x-1)\tilde{H}(x-1) + \tilde{G}(x)\tilde{H}(x) = 1,\ x\in[0,1]\setminus(\tilde{V}+1). \tag{5.14}$$

Recall that $\tilde{G}(x-1), \tilde{G}(x), \tilde{H}(x-1)$ are differentiable on $[0,1]\setminus(\tilde{V}+1)$. By (5.13), the same is the case for $\tilde{H}(x)$. Since $\operatorname{supp}\tilde{H}\subseteq[-1,1]$, it remains to show that $\tilde{H}$ is differentiable on $\tilde{V}+1 = \{0\}\bigcup[a+1,b+1]\bigcup\{c\}\bigcup\{1\}$, and that $D_-\tilde{H}(1)=0$. Note that $\tilde{G}$ is differentiable on $\{0\}\bigcup[a+1,b+1]$. Thus, by (5.7) and (5.12), it suffices to show

$$D_+\tilde{H}(0) = D\left(\frac{1}{\tilde{G}}\right)(0); \tag{5.15}$$

$$D_-\tilde{H}(a+1) = D\left(\frac{1}{\tilde{G}}\right)(a+1);\ D_+\tilde{H}(b+1) = D\left(\frac{1}{\tilde{G}}\right)(b+1); \tag{5.16}$$

$$D_-\tilde{H}(c) = D_+\tilde{H}(c),\ D_-\tilde{H}(1) = 0. \tag{5.17}$$

We first show (5.15). From (5.7), we get

$$\tilde{H}(-1) = 0,\ \tilde{G}(0)\tilde{H}(0) = 1. \tag{5.18}$$

Putting this into (5.14), we have for $x\in[0,1]\setminus(\tilde{V}+1)$,

$$\tilde{G}(x-1)\left(\tilde{H}(x-1)-\tilde{H}(-1)\right) + \tilde{G}(x)\left(\tilde{H}(x)-\tilde{H}(0)\right) + \tilde{H}(0)\left(\tilde{G}(x)-\tilde{G}(0)\right) = 0,$$

or by dividing with x,

$$\tilde{G}(x-1)\left(\frac{\tilde{H}(x-1)-\tilde{H}(-1)}{x}\right) + \tilde{G}(x)\left(\frac{\tilde{H}(x)-\tilde{H}(0)}{x}\right) + \tilde{H}(0)\left(\frac{\tilde{G}(x)-\tilde{G}(0)}{x}\right) = 0.$$

This implies that

$$\frac{\tilde{H}(x)-\tilde{H}(0)}{x} = -\frac{\tilde{G}(x-1)\left(\frac{\tilde{H}(x-1)-\tilde{H}(-1)}{x}\right) + \tilde{H}(0)\left(\frac{\tilde{G}(x)-\tilde{G}(0)}{x}\right)}{\tilde{G}(x)},\ x\in[0,1]\setminus(\tilde{V}+1).$$

Using (5.9), it follows that

$$D_+\tilde{H}(0) = -\frac{\tilde{G}((-1)^+)D_+\tilde{H}(-1) + \tilde{H}(0)D_+\tilde{G}(0)}{\tilde{G}(0^+)} = -\frac{\tilde{H}(0)D_+\tilde{G}(0)}{\tilde{G}(0^+)}.$$

Since $\tilde{G}(x)$ is differentiable at $0 \notin \tilde{W}$, we have $\tilde{G}(0^+) = \tilde{G}(0)$, $D_+\tilde{G}(0) = D\tilde{G}(0)$. This together with (5.18) implies that

$$D_+\tilde{H}(0) = -\frac{\tilde{H}(0)D\tilde{G}(0)}{\tilde{G}(0)} = -\frac{D\tilde{G}(0)}{\tilde{G}^2(0)} = D\left(\frac{1}{\tilde{G}}\right)(0).$$

This proves that (5.15) holds. The results in (5.16) and (5.17) can be shown in a similar way. □

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