On the duality principle by Casazza, Kutyniok, and Lammers *

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September 12, 2010

Abstract

The R-dual sequences of a frame $\{f_i\}_{i\in I}$, introduced by Casazza, Kutyniok and Lammers in [1], provide a powerful tool in the analysis of duality relations in general frame theory. In this paper we derive conditions for a sequence $\{\omega_j\}_{j\in I}$ to be an R-dual of a given frame $\{f_i\}_{i\in I}$. In particular we show that the R-duals $\{\omega_j\}_{j\in I}$ can be characterized in terms of frame properties of an associated sequence $\{n_i\}_{i\in I}$. We also derive the duality results obtained for tight Gabor frames in [1] as a special case of a general statement for R-duals of frames in Hilbert spaces. Finally we consider a relaxation of the R-dual setup of independent interest. Several examples illustrate the results.

Math Subject Classifications: 42C15, 42C40, 42A38.
Keywords: Duality principle, Frame, Riesz basis, Gabor system, Wexler-Raz theorem.

1 Introduction and notation

Let $\{f_i\}_{i\in I}$ denote a frame for a separable Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. In [1], Casazza, Kutyniok, and Lammers introduced the *Riesz-dual*

*This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0007614).

sequence (R-dual sequence) *of* $\{f_i\}_{i\in I}$ *with respect to a choice of orthonormal bases* $\{e_i\}_{i\in I}$ *and* $\{h_i\}_{i\in I}$ as the sequence $\{\omega_j\}_{j\in I}$ given by

$$\omega_j = \sum_{i\in I} \langle f_i, e_j\rangle h_i, \ j \in I. \tag{1.1}$$

The paper [1] demonstrates that there is a strong relationship between the frame-theoretic properties of $\{\omega_j\}_{j\in I}$ and $\{f_i\}_{i\in I}$, see Theorem 1.3 below for details. The purpose of this paper is to analyze the concept of R-dual sequence from another angle than it was done in [1]. Technically this is done by considering a dual formulation of (1.1), namely, for a given frame $\{f_i\}_{i\in I}$ and a (Riesz) sequence $\{\omega_j\}_{j\in I}$ to search for orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ such that

$$f_i = \sum_{j\in I} \langle \omega_j, h_i\rangle e_j, \ i \in I. \tag{1.2}$$

Using this approach we state a number of equivalent conditions for $\{\omega_j\}_{j\in I}$ to be an R-dual of $\{f_i\}_{i\in I}$. In particular we introduce a sequence $\{n_i\}_{i\in I}$ that can be used to check whether $\{\omega_j\}_{j\in I}$ is an R-dual of $\{f_i\}_{i\in I}$ or not; in fact, the answer is yes if and only if $\{n_i\}_{i\in I}$ is a tight frame sequence with frame bound $E = 1$.

One of the key properties of the R-duals is a certain duality relation that resembles the duality principle in Gabor analysis. The driving force in the article [1] was the question whether the duality principle in Gabor analysis actually can be derived from the theory of the R-duals. The question remains unsolved, but in [1] a positive conclusion is derived in the special case of a tight Gabor frame. The results presented here shed new light on this issue: in fact, the partial result in [1] turns out to be a consequence of a general result about R-duals, valid for any tight frame in any Hilbert space.

In the rest of this section we review some of the needed facts about the R-duals, as well as tools from frame theory. We also state a few basic results about Gabor systems and their relationship to the R-dual concept. Our main results for the R-duals associated with general frames are stated in Section 2. Section 3 deals with an relaxation of the above setup: we show that for the relevant sequences $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ and any orthonormal basis $\{e_i\}_{i\in I}$ we can always find an *orthogonal system* $\{h_i\}_{i\in I}$ such that (1.2) holds. An additional condition on the relationship between $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ implies that $\{h_i\}_{i\in I}$ can even be chosen as an orthonormal system, i.e., compared to the general agenda only the completeness of $\{h_i\}_{i\in I}$ is missing. Appendix A contains a proof of a technical lemma.

Frames and Riesz bases. It will be essential to distinguish carefully between sequences forming a basis/frame for the entire Hilbert space $\mathcal{H}$ or a subspace thereof. For that reason we begin with the following standard definition:

Definition 1.1 *Let I denote a countable index set.*

(i) A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\,||f||^2, \ \forall f \in \mathcal{H}.$$

(ii) A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\,||f||^2, \ \forall f \in \mathcal{H}.$$

The numbers A, B are called frame bounds. The frame is tight if we can choose $A = B$.

(iii) A sequence $\{\omega_j\}_{j\in I}$ in $\mathcal{H}$ is a Riesz sequence if there exist constants $C, D > 0$ such that

$$C\sum_{j\in I} |c_j|^2 \leq \left|\left|\sum_{j\in I} c_j\omega_i\right|\right|^2 \leq D\sum_{j\in I} |c_j|^2$$

for all finite sequences $\{c_i\}_{i\in I}$. The numbers C, D are called (Riesz) bounds.

(iv) A Riesz sequence $\{\omega_j\}_{j\in I}$ is a Riesz basis for $\mathcal{H}$ if $\overline{span}\{\omega_j\}_{j\in I} = \mathcal{H}$.

Given any sequence $\{\omega_j\}_{j\in I}$ in $\mathcal{H}$, let

$$W := \overline{\text{span}}\{\omega_j\}_{j\in I}.$$

In case $\{\omega_j\}_{j\in I}$ is a Riesz sequence, it is well known that $\{\omega_j\}_{j\in I}$ has a unique *dual Riesz sequence* belonging to W: that is, there exists a unique Riesz sequence $\{\widetilde{\omega_k}\}_{k\in I}$ of elements in W such that

$$\langle \omega_j, \widetilde{\omega_k}\rangle = \delta_{j,k},\ \ j,k \in I. \tag{1.3}$$

If $\{\omega_j\}_{j\in I}$ has Riesz bounds C, D, then the dual Riesz sequence has bounds $1/D, 1/C$.

Recall that the sequence $\{\omega_j\}_{j\in I}$ has *infinite deficit* if

$$\dim(\overline{\operatorname{span}}\{\omega_j\}_{j\in I}^{\perp}) = \infty.$$

The R-duals of a sequence $\{f_i\}_{i\in I}$. We now state the definition of the R-dual sequence, repeated from [1]. We are only interested in the case where $\mathcal{H}$ is infinite-dimensional, in which case we can also index the R-duals by I:

Definition 1.2 *Let $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ denote orthonormal bases for $\mathcal{H}$, and let $\{f_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ for which*

$$\sum_{i\in I} |\langle f_i, e_j\rangle|^2 < \infty,\ \ \forall j \in I. \tag{1.4}$$

The R-dual of $\{f_i\}_{i\in I}$ with respect to the orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ is the sequence $\{\omega_j\}_{j\in I}$ given by

$$\omega_j = \sum_{i\in I} \langle f_i, e_j\rangle h_i,\ \ j \in I. \tag{1.5}$$

Note that any given sequence $\{f_i\}_{i\in I}$ has *many* associated R-dual sequences, namely, one for each choice of the orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$. We collect the main results about the relationship between $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ from [1].

Theorem 1.3 *Let $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ denote orthonormal bases for $\mathcal{H}$, and let $\{f_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ for which $\sum_{i\in I} |\langle f_i, e_j\rangle|^2 < \infty$ for all $j \in I$. Define the R-dual $\{\omega_j\}_{j\in I}$ as in* (1.5)*. Then the following hold:*

(i) For all $i \in I$,

$$f_i = \sum_{j\in I} \langle \omega_j, h_i\rangle e_j, \tag{1.6}$$

i.e., $\{f_i\}_{i\in I}$ is the R-dual sequence of $\{\omega_j\}_{j\in I}$ w.r.t. the orthonormal bases $\{h_i\}_{i\in I}$ and $\{e_i\}_{i\in I}$.

(ii) $\{f_i\}_{i\in I}$ *is a Bessel sequence if and only* $\{\omega_i\}_{i\in I}$ *is a Bessel sequence; the Bessel bounds coincide.*

(iii) $\{f_i\}_{i\in I}$ *satisfies the lower frame condition with bound* A *if and only if* $\{\omega_j\}_{j\in I}$ *satisfies the lower Riesz sequence condition with bound* A.

(iv) $\{f_i\}_{i\in I}$ *is a frame for* $\mathcal{H}$ *with bounds* A, B *if and only if* $\{\omega_j\}_{j\in I}$ *is a Riesz sequence in* $\mathcal{H}$ *with bounds* A, B.

(v) Two Bessel sequences $\{f_i\}_{i\in I}$ *and* $\{g_i\}_{i\in I}$ *in* $\mathcal{H}$ *are dual frames if and only if the associated R-dual sequences* $\{\omega_j\}_{j\in I}$ *and* $\{\gamma_j\}_{j\in I}$ *w.r.t. the same choices of orthonormal bases* $\{e_i\}_{i\in I}$ *and* $\{h_i\}_{i\in I}$ *satisfy*

$$\langle \omega_j, \gamma_k \rangle = \delta_{j,k}, \ \ j, k \in I. \tag{1.7}$$

The property in Theorem 1.3(v) is a key result and the main motivation for the interest in the R-dual. The next paragraph explains this in more detail.

Gabor systems. For a function $g \in L^2(\mathbb{R})$, the *Gabor system* associated with g and two given parameters a, b is the collection of functions given by

$$\{e^{2\pi imbx} g(x - na)\}_{m,n\in\mathbb{Z}}.$$

We will use the short notation $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to denote the Gabor system.

The *duality principle* is one of the most fundamental results in Gabor analysis. It was discovered almost simultaneously by three groups of researchers: Janssen [6], Daubechies, Landau, and Landau [3], and Ron and Shen [7]. The duality principle concerns the relationship between frame properties for a function g with respect to the lattice $\{(na, mb)\}_{m,n\in\mathbb{Z}}$ and with respect to the so-called *dual lattice* $\{(n/b, m/a)\}_{m,n\in\mathbb{Z}}$:

Theorem 1.4 *Let* $g \in L^2(\mathbb{R})$ *and* $a, b > 0$ *be given. Then the Gabor system* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *with bounds* A, B *if and only if* $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *is a Riesz sequence with bounds* A, B.

Comparing Theorem 1.4 with Theorem 1.3(iv) makes it natural to ask whether $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ can be realized as the R-dual of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with respect to appropriate choices of orthonormal bases $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ and $\{h_{m,n}\}_{m,n\in\mathbb{Z}}$. Combined with Theorem 1.3(v), the well known *Wexler-Raz theorem* provides strong support for this hypothesis:

Theorem 1.5 *If the Gabor systems* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ *are dual frames, then the Gabor systems* $\{\frac{1}{\sqrt{ab}}\,E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *and* $\{\frac{1}{\sqrt{ab}}\,E_{m/a}T_{n/b}h\}_{m,n\in\mathbb{Z}}$ *are biorthogonal.*

In [1], Casazza, Kutyniok and Lammers proved the following partial result:

Theorem 1.6 *Assuming that* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *the following hold:*

(i) If $ab = 1$, *then* $\{\frac{1}{\sqrt{ab}}\,E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *can be realized as the R-dual of* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *w.r.t. certain choices of orthonormal bases* $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ *and* $\{h_{m,n}\}_{m,n\in\mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

(ii) If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a tight frame, then* $\{\frac{1}{\sqrt{ab}}\,E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *can be realized as the R-dual of* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *w.r.t. certain choices of orthonormal bases* $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ *and* $\{h_{m,n}\}_{m,n\in\mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

Among other results, we will show that Theorem 1.6(ii) is a consequence of a general result that is valid for any tight frame in any separable Hilbert space.

2 Duality for general frames

Our first goal is to find conditions on two sequences $\{f_i\}_{i\in I}, \{\omega_j\}_{j\in I}$ such that $\{\omega_j\}_{j\in I}$ is the R-dual of $\{f_i\}_{i\in I}$ with respect to *some* choice of the orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$. Assume that $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$. By Theorem 1.3 this implies that any R-dual sequence $\{\omega_j\}_{j\in I}$ is a Riesz sequence in $\mathcal{H}$ and that (1.6) holds. On the other hand, Theorem 1.3 shows that if $\{\omega_j\}_{j\in I}$ is a Riesz sequence and (1.6) holds, then $\{\omega_j\}_{j\in I}$ is a R-dual of $\{f_i\}_{i\in I}$. Thus we arrive at the following key question:

Question: Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ and $\{\omega_j\}_{j\in I}$ a Riesz sequence in $\mathcal{H}$. Under what conditions can we find orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ for $\mathcal{H}$ such that (1.6) holds?

We first show that for any Riesz sequence $\{\omega_j\}_{j\in I}$, any sequence $\{f_i\}_{i\in I}$, and any orthonormal basis $\{e_i\}_{i\in I}$, we can actually find and characterize the sequences $\{h_i\}_{i\in I}$ for which (1.6) holds; thus, the remaining question is whether at least one of these sequences forms an orthonormal basis for $\mathcal{H}$. The key point in the analysis is the definition of a sequence $\{n_i\}_{i\in I}$, given by

$$n_i := \sum_{k\in I} \langle e_k, f_i\rangle \widetilde{\omega_k}, \; i \in I, \tag{2.1}$$

where $\{\widetilde{\omega_k}\}_{k\in I}$ is the dual Riesz sequence of $\{\omega_j\}_{j\in I}$. Note that under the above assumptions the sequences $\{\widetilde{\omega_k}\}_{k\in I}$ and $\{e_i\}_{i\in I}$ are Bessel sequences, implying that the infinite series defining n_i is convergent.

Note that while the motivation for our analysis comes from the case where $\{f_i\}_{i\in I}$ is a frame, several of our results hold for any sequence $\{f_i\}_{i\in I}$. Thus, we only state the frame assumption when it is necessary. We begin with a simple lemma, relating the involved sequences:

Lemma 2.1 *Let $\{\omega_j\}_{j\in I}$ be a Riesz basis for the subspace W of $\mathcal{H}$, with dual Riesz basis $\{\widetilde{\omega_k}\}_{k\in I}$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Given any sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$, define $\{n_i\}_{i\in I}$ as in (2.1). Then*

$$\langle \omega_j, n_i\rangle = \langle f_i, e_j\rangle, \; \forall i, j \in I.$$

Lemma 2.1 is a direct consequence of the definition of n_i and (1.3). Our starting point is now to characterize the sequences $\{h_i\}_{i\in I}$ for which (1.6) holds:

Proposition 2.2 *Let $\{\omega_j\}_{j\in I}$ be a Riesz basis for the subspace W of $\mathcal{H}$, with dual Riesz basis $\{\widetilde{\omega_k}\}_{k\in I}$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Given any sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$, the following hold:*

(i) There exists a sequence $\{h_i\}_{i\in I}$ in $\mathcal{H}$ such that

$$f_i = \sum_{j\in I} \langle \omega_j, h_i\rangle e_j, \; \forall i \in I. \tag{2.2}$$

(ii) The sequences $\{h_i\}_{i\in I}$ satisfying (2.2) are characterized as

$$h_i = m_i + n_i, \tag{2.3}$$

where n_i is given by (2.1) and $m_i \in W^{\perp}$.

(iii) If $\{\omega_j\}_{j\in I}$ is a Riesz basis for $\mathcal{H}$, then (2.2) has the unique solution

$$h_i = n_i, \; i \in I.$$

Proof. Expanding f_i in the orthonormal basis $\{e_j\}_{j\in I}$ and using Lemma 2.1,

$$f_i = \sum_{j\in I} \langle f_i, e_j\rangle e_j = \sum_{j\in I} \langle \omega_j, n_i\rangle e_j, \ i \in I,$$

i.e., the choice $h_i = n_i$ satisfies (2.2). This proves (i). For $m_i \in W^\perp$ it now follows from $\omega_j \in W$ that the choice $h_i = m_i + n_i$ will satisfy (2.2) as well. In order to complete the proof of (ii) we only need to show that all solutions $\{h_i\}_{i\in I}$ of (2.2) are of the form in (2.3). Let $\{h_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ satisfying (2.2). Fix any $i \in I$. We can write $h_i = m_i + n_i$ with $m_i := h_i - n_i$. The expansion coefficients of f_i in terms of the basis $\{e_i\}_{i\in I}$ are unique, so from

$$f_i = \sum_{j\in I} \langle \omega_j, h_i\rangle e_j = \sum_{j\in I} \langle \omega_j, n_i\rangle e_j$$

it follows that

$$\langle \omega_j, h_i\rangle = \langle \omega_j, n_i\rangle, \ \forall j \in I,$$

i.e.,

$$\langle \omega_j, m_i\rangle = 0, \ \forall j \in I.$$

This implies that $m_i \in W^\perp$. This proves (ii). The result in (iii) is a consequence of (ii). □

With Proposition 2.2 at hand our goal is now to find conditions under which an orthonormal basis $\{h_i\}_{i\in I}$ for $\mathcal{H}$ of the form (2.3) exists. We note that Proposition 2.2 did not use any assumption on $\{f_i\}_{i\in I}$ or any relationship between $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$. The uniqueness statement in Proposition 2.2(iii) makes it easy to find a case where no orthonormal basis of the form (2.3) exists, even if we assume that $\{f_i\}_{i\in I}$ is a frame; for example let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$, let $\{\omega_i\}_{i\in I} := \{e_i\}_{i\in I}$, and take $\{f_i\}_{i\in I} := \{2e_1, e_2, e_3, \cdots\}$. Then a simple calculation shows that the only solution of (2.3) is $h_1 = 2e_1, \ h_i = e_i, \ i \geq 2$.

We will now have a closer look at the properties of the sequence $\{n_i\}_{i\in I}$ in (2.1).

Lemma 2.3 *Let $\{\omega_j\}_{j\in I}$ be a Riesz sequence in $\mathcal{H}$ with bounds C, D, and let $\{e_i\}_{i\in I}$ an orthonormal basis for $\mathcal{H}$. Given a frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$ with frame bounds A, B, the sequence $\{n_i\}_{i\in I}$ in* (2.1) *is a frame for $W := \overline{span}\{\omega_j\}_{j\in I}$ with frame bounds $A/D, B/C$.*

Proof. It is clear that $n_i \in W,\ \forall i \in I$. Now, for any $f \in W$,

$$\begin{aligned}
\sum_{i\in I} |\langle f, n_i\rangle|^2 &= \sum_{i\in I} \left| \langle f, \sum_{k\in I} \langle e_k, f_i\rangle \widetilde{\omega_k}\rangle \right|^2 \\
&= \sum_{i\in I} \left| \sum_{k\in I} \langle f, \widetilde{\omega_k}\rangle \langle f_i, e_k\rangle \right|^2 \\
&= \sum_{i\in I} \left| \langle f_i, \sum_{k\in I} \langle \widetilde{\omega_k}, f\rangle e_k\rangle \right|^2 .
\end{aligned}$$

Note that $\{\widetilde{\omega_k}\}_{k\in I}$ is a Riesz basis for W with bounds $1/D, 1/C$. Thus the above calculation yields that

$$\begin{aligned}
\sum_{i\in I} |\langle f, n_i\rangle|^2 \geq A \left\| \sum_{k\in I} \langle \widetilde{\omega_k}, f\rangle e_k \right\|^2 &= A \sum_{k\in I} |\langle \widetilde{\omega_k}, f\rangle|^2 \\
&\geq \frac{A}{D} ||f||^2.
\end{aligned}$$

The proof for the upper bound is similar. □

We will now present a solution to our key question, i.e., characterize the existence of an orthonormal basis $\{h_i\}_{i\in I}$ for $\mathcal{H}$ such that (2.2) holds. We note that the case where the Riesz sequence $\{\omega_j\}_{j\in I}$ spans the entire space $\mathcal{H}$ is solved in Proposition 2.2(iii). Thus, we concentrate on the case where the Riesz sequence $\{\omega_j\}_{j\in I}$ spans a proper subspace of $\mathcal{H}$.

Theorem 2.4 *Let $\{\omega_j\}_{j\in I}$ be a Riesz sequence spanning a proper subspace W of $\mathcal{H}$ and $\{e_i\}_{i\in I}$ an orthonormal basis for $\mathcal{H}$. Given any frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$, the following are equivalent:*

(i) $\{\omega_j\}_{j\in I}$ is an R-dual of $\{f_i\}_{i\in I}$ w.r.t. $\{e_i\}_{i\in I}$ and some orthonormal basis $\{h_i\}_{i\in I}$.

(ii) There exists an orthonormal basis $\{h_i\}_{i\in I}$ *for* $\mathcal{H}$ *satisfying* (2.2).

(iii) The sequence $\{n_i\}_{i\in I}$ *in* (2.1) *is a tight frame for* W *with frame bound* $E = 1$, *i.e., a Parseval frame.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from Proposition 2.2.
(ii) $\Rightarrow$(iii). Let P denote the orthogonal projection of $\mathcal{H}$ onto W. The expression in (2.3) for all solutions to (2.2) shows that a sequence $\{h_i\}_{i\in I}$ in $\mathcal{H}$ is a solution if and only if $Ph_i = n_i,\ \forall i \in I$. Now, it is well known that the projection of an orthonormal basis onto a subspace forms a tight frame for that subspace with frame bound equal to one. Thus, if $\{h_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$, then necessarily $\{n_i\}_{i\in I}$ is a tight frame for W with frame bound $E = 1$.
(iii) $\Rightarrow$(ii). If $\{n_i\}_{i\in I}$ is a tight frame for W with frame bound $E = 1$, then Naimark's theorem (see, e.g., [5]) says that there exists an orthonormal basis for a larger Hilbert space such that $Ph_i = n_i$. Since W is assumed to be a proper subspace of $\mathcal{H}$ we can identify the larger Hilbert space with $\mathcal{H}$, which leads to the desired conclusion. $\square$

Using Theorem 2.4 we can now give an example of a frame $\{f_i\}_{i\in I}$ and a Riesz sequence $\{\omega_j\}_{j\in I}$ that can not be an R-dual of $\{f_i\}_{i\in I}$ w.r.t. a given orthonormal basis $\{e_i\}_{i\in I}$ and any choice of $\{h_i\}_{i\in I}$, despite the fact that the bounds for $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ coincide:

Example 2.5 Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$ and

$$\{f_i\}_{i\in I} := \{2e_1, e_1, e_2, e_3, \dots\},$$

$$\{\omega_j\}_{j\in I} = \{5e_1, e_3, e_5, \dots\}.$$

Then $\{f_i\}_{i\in I}$ is a frame with bounds $A = 1, B = 5$, and $\{\omega_j\}_{j\in I}$ is a Riesz sequence with the same bounds. The dual Riesz sequence is

$$\{\widetilde{\omega_k}\}_{k\in I} = \{\frac{1}{5}e_1, e_3, e_5, \dots\}.$$

Direct calculation shows that

$$\{n_i\}_{i\in I} = \{\frac{2}{5}e_1, \frac{1}{5}e_1, e_3, e_5, \dots\}.$$

The frame is clearly not tight, so $\{\omega_j\}_{j\in I}$ is not an R-dual of $\{f_i\}_{i\in I}$ with respect to $\{e_i\}_{i\in I}$ and any choice of an orthonormal basis $\{h_i\}_{i\in I}$. $\square$

Combining Lemma 2.3 and Theorem 2.4, we obtain a partial answer to our key question. Note that the assumptions stated in the following result also can be formulated by saying that $\{\omega_j\}_{j\in I}$ is an equal norm orthogonal sequence.

Corollary 2.6 *Assume that $\{\omega_j\}_{j\in I}$ is a Riesz sequence with upper and lower bound A, spanning a proper subspace of $\mathcal{H}$, and that $\{f_i\}_{i\in I}$ is a tight frame for $\mathcal{H}$ with frame bound A. Then $\{\omega_i\}_{i\in I}$ is an R-dual of $\{f_i\}_{i\in I}$.*

Proof. The assumptions imply by Lemma 2.3 that $\{n_i\}_{i\in I}$ is a tight frame for W with frame bound $E = 1$, for any choice of the orthonormal basis $\{e_i\}_{i\in I}$. Now the result follows from Theorem 2.4. □

The assumptions in Corollary 2.6 correspond exactly to the known relationship between a tight Gabor frame and the corresponding Gabor system on the dual lattice. Thus Corollary 2.6 is a generalization of the result from [1] that we stated in Theorem 1.6(ii).

The assumption that $\{\omega_j\}_{j\in I}$ spans a proper subspace of $\mathcal{H}$ is essential in Corollary 2.6:

Example 2.7 Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$, and let

$$\begin{aligned} \{f_i\}_{i\in\mathbb{N}} &:= \{e_1, e_1, e_2, e_2, \dots\}, \\ \{\omega_j\}_{j\in\mathbb{N}} &:= \{e_1, e_2, \cdots\}. \end{aligned}$$

Then $\{f_i\}_{i\in\mathbb{N}}$ is a tight frame for $\mathcal{H}$, but $\{\omega_j\}_{j\in\mathbb{N}}$ is not an R-dual w.r.t. $\{e_i\}_{i\in\mathbb{N}}$ and any choice of $\{h_i\}_{i\in\mathbb{N}}$. In fact, if $\{\omega_j\}_{j\in\mathbb{N}}$ was an R-dual of $\{f_i\}_{i\in\mathbb{N}}$ with respect to $\{e_i\}_{i\in\mathbb{N}}$ and some orthonormal basis $\{h_i\}_{i\in\mathbb{N}}$, the definition (1.5) with $j = 1$ would show that $e_1 = h_1 + h_2$, which is impossible. □

With Theorem 2.4 and Corollary 2.6 in mind it is natural to ask whether an orthonormal basis $\{h_i\}_{i\in I}$ for $\mathcal{H}$ satisfying (2.2) can be found if the frame $\{f_i\}_{i\in I}$ is non-tight. Intuitively this sounds unlikely - but there are cases where the answer is yes:

Example 2.8 Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$, and define the sequences $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ by

$$\{f_i\}_{i\in I} = \{\frac{1}{2}e_1, e_2, e_3, \cdots\},$$

respectively,

$$\{\omega_j\}_{j\in I} = \{\frac{1}{2}e_1, e_2, e_3, \cdots\}.$$

Then

$$\widetilde{\omega_k} = \{2e_1, e_2, e_3, \cdots\},$$

and thus

$$n_i = \sum_{k\in I} \langle e_k, f_i\rangle \widetilde{\omega_k} = e_i, \ \forall i \in I.$$

Thus $\{n_i\}_{i\in I}$ is an orthonormal basis and therefore tight, despite the fact that $\{f_i\}_{i\in I}$ is non-tight. $\square$

Theorem 2.4 leads to a simple criterion for $\{\omega_j\}_{j\in I}$ to be an R-dual of $\{f_i\}_{i\in I}$. The result can be considered as an if and only if version of Proposition 5 in [1]:

Corollary 2.9 *Let $\{\omega_j\}_{j\in I}$ be a Riesz basis for the subspace W of $\mathcal{H}$ and let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. For any $c = \{c_i\}_{i\in I} \in \ell^2(I)$, let the vectors e_c and ω_c be related by*

$$e_c = \sum_{j\in I} \overline{c_j} e_j, \quad \omega_c = \sum_{j\in I} c_j \omega_j. \tag{2.4}$$

Then $\{\omega_j\}_{j\in I}$ is an R-dual of $\{f_i\}_{i\in I}$ w.r.t. $\{e_i\}_{i\in I}$ and some orthonormal basis $\{h_i\}_{i\in I}$ if and only if

$$\sum_{i\in I} |\langle f_i, e_c\rangle|^2 = ||\omega_c||^2$$

for all choices of the sequence $c \in \ell^2(I)$.

Proof. Let $\{\widetilde{\omega_k}\}_{k\in I}$ be the dual Riesz basis of $\{\omega_j\}_{j\in I}$ and define $\{n_i\}_{i\in I}$ as in (2.1). By the result in Lemma 2.1 and the relation between e_c and ω_c,

$$\langle n_i, \omega_c\rangle = \sum_{j\in I} \overline{c_j} \langle n_i, \omega_j\rangle = \sum_{j\in I} \overline{c_j} \langle e_j, f_i\rangle = \langle e_c, f_i\rangle.$$

Thus

$$\sum_{i\in I} |\langle n_i, \omega_c\rangle|^2 = \sum_{i\in I} |\langle e_c, f_i\rangle|^2.$$

The result now follows from Theorem 2.4. $\square$

3 Orthonormal sequences $\{h_i\}_{i\in I}$

In Proposition 2.2 we have shown that a Riesz sequence $\{\omega_j\}_{j\in I}$ is an R-dual of a frame $\{f_i\}_{i\in I}$ if there exists orthonormal bases $\{h_i\}_{i\in I}$ and $\{e_i\}_{i\in I}$ such that

$$f_i = \sum_{j\in I} \langle \omega_j, h_i\rangle e_j, \ \forall i \in I. \tag{3.1}$$

In order to gain further insight into the problem we will now consider a weaker version of this condition. In fact, we will assume that $\{e_i\}_{i\in I}$ is a given orthonormal basis, and ask for the existence of an *orthogonal,* resp. *orthonormal* sequence $\{h_i\}_{i\in I}$ such that (3.1) holds. We will show that these questions have very general answers.

We begin with a lemma, stating an observation of independent interest. For the proof, see Appendix A.

Lemma 3.1 *Assume that $\{f_i\}_{i\in I}$ is a Bessel sequence with bound B. Then for any f_i, f_j,*

$$|\langle f_i, f_j\rangle|^2 \le B\left(B - ||f_i||^2 - ||f_j||^2\right) + ||f_i||^2||f_j||^2. \tag{3.2}$$

Note that the result in Lemma 3.1 is trivial if $B - ||f_i||^2 - ||f_j||^2 \ge 0$. However, under the assumptions given here it can very well happen that $B - ||f_i||^2 - ||f_j||^2 < 0$, and for such elements f_i, f_j the result is an improvement of Cauchy–Schwarz' inequality.

Theorem 3.2 *Let $\{\omega_j\}_{j\in I}$ be a Riesz sequence in $\mathcal{H}$ having infinite deficit, and let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Then the following hold:*

(i) For any sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ there exists an orthogonal sequence $\{h_i\}_{i\in I}$ in $\mathcal{H}$ such that

$$f_i = \sum_{j\in I} \langle \omega_j, h_i\rangle e_j, \ \forall i \in I. \tag{3.3}$$

(ii) Assume that $\{f_i\}_{i\in I}$ is a Bessel sequence with bound B and that $\{\omega_j\}_{j\in I}$ has a lower Riesz basis bound $C \ge B$. Then there exists an orthonormal sequence $\{h_i\}_{i\in I}$ such that (3.3) *holds.*

(iii) For any Bessel sequence $\{f_i\}_{i\in I}$ and regardless of the lower Riesz bound for $\{\omega_j\}_{j\in I}$, there exist an orthonormal sequence $\{h_i\}_{i\in I}$ in $\mathcal{H}$ and a constant $\alpha > 0$ such that

$$f_i = \sum_{j\in I} \langle \alpha\omega_j, h_i \rangle e_j, \ \forall i \in I. \tag{3.4}$$

Proof. The proof of (i) is based on Proposition 2.2. We consider again the vectors n_i in (2.1) and want to find $m_i \in W^\perp$, $i \in I$, such that $h_i := m_i + n_i$ is an orthogonal sequence. For notational convenience, assume that $I = \mathbb{N}$. Note that with such a choice of h_i, we know that (3.3) is satisfied. Note also that

$$\langle h_i, h_j \rangle = \langle n_i, n_j \rangle + \langle m_i, m_j \rangle, \ \forall i, j \in \mathbb{N}. \tag{3.5}$$

We will use the following inductive procedure. Choose $m_1 \in W^\perp$ arbitrarily. Now, take $m_2 \in W^\perp$ such that

$$\langle h_1, h_2 \rangle = 0,$$

i.e., such that

$$\langle m_1, m_2 \rangle = -\langle n_1, n_2 \rangle.$$

In general, assuming that we have constructed $m_1, \dots, m_N \in W^\perp$ such that $\{h_i\}_{i=1}^N$ is an orthogonal system, take $m_{N+1} \in W^\perp$ such that

$$\langle h_k, h_{N+1} \rangle = 0, \ k = 1, \dots, N,$$

i.e., such that

$$\langle m_k, m_{N+1} \rangle = -\langle n_k, n_{N+1} \rangle, \ k = 1, \dots, N.$$

This can always be done because $\{\omega_j\}_{j\in I}$ is assumed to have infinite deficit. We conclude that $\{h_i\}_{i\in I}$ forms an orthogonal system, as desired.

For the proof of (ii), let B denote an upper frame bound for $\{f_i\}_{i\in I}$ and C a lower bound for the Riesz sequence $\{\omega_j\}_{j\in I}$. By an argument like in the proof of Lemma 2.3, the sequence $\{n_i\}_{i\in I}$ is a Bessel sequence with bound $\frac{B}{C} \le 1$; in particular, the norms of the vectors n_i are uniformly bounded by $||n_i|| \le 1$. We now aim at a construction of a sequence $\{h_i\}_{i\in I}$ satisfying

(3.3) and $||h_i|| = 1,\ \forall i \in I$. We use the inductive procedure outlined in (i), but now paying attention to the norm of the vectors h_i. First we choose $m_1 \in W^\perp$ such that $||h_1|| = 1$, i.e., such that

$$||m_1|| = \sqrt{1 - ||n_1||^2}.$$

We now want to choose $m_2 \in W^\perp$ such that $||h_2|| = 1$ and $\langle h_1, h_2\rangle = 0$; this means that we want that

$$||m_2|| = \sqrt{1 - ||n_2||^2} \ \text{ and } \ \langle m_1, m_2\rangle = -\langle n_1, n_2\rangle. \tag{3.6}$$

The first condition in (3.6) can always be satisfied; and the second can be satisfied for a sequence m_2 with $||m_2|| = \sqrt{1 - ||n_2||^2}$ if and only if

$$\sqrt{1 - ||n_1||^2}\sqrt{1 - ||n_2||^2} \geq |\langle n_1, n_2\rangle|. \tag{3.7}$$

The condition in (3.7) is satisfied by Lemma 3.1.

Following the inductive procedure outlined in (i), we see that it is possible to construct an orthonormal sequence $\{h_i\}_{i\in I}$ satisfying (3.3) if

$$\sqrt{1 - ||n_i||^2}\sqrt{1 - ||n_j||^2} \geq |\langle n_i, n_j\rangle|,\ \forall i, j \in I,$$

which is satisfied by Lemma 3.1.

Finally, the result in (iii) is obtained by scaling of the Riesz sequence $\{\omega_j\}_{j\in I}$ in such a way that we obtain a sequence $\{\alpha\,\omega_j\}_{j\in I}$ to which we can apply (ii). □

4 Appendix A - proof of Lemma 3.1

Proof of Lemma 3.1: We give the proof for the case $B = 1$; the general case follows from here by replacing $\{f_i\}_{i\in I}$ by $\{f_i/\sqrt{B}\}_{i\in I}$. For notational convenience we take $i = 1, j = 2$.

First, we assume $\langle f_1, f_2\rangle$ is real. Let $f := xf_1 + f_2$ for some $x \in \mathbb{R}$. Then

$$||f||^2 = x^2||f_1||^2 + 2x\langle f_1, f_2\rangle + ||f_2||^2 \tag{4.1}$$

and

$$\begin{aligned} |\langle f, f_1\rangle|^2 + |\langle f, f_2\rangle|^2 &= ||f_1||^4x^2 + 2\langle f_1, f_2\rangle||f_1||^2x + |\langle f_1, f_2\rangle|^2 \\ &+ |\langle f_1, f_2\rangle|^2x^2 + 2\langle f_1, f_2\rangle||f_2||^2x + ||f_2||^4 \\ &= (||f_1||^4 + |\langle f_1, f_2\rangle|^2)x^2 + 2\langle f_1, f_2\rangle(||f_1||^2 + ||f_2||^2)x \\ &+ ||f_2||^4 + |\langle f_1, f_2\rangle|^2 \end{aligned} \tag{4.2}$$

Using the upper frame condition on f,

$$\sum_{i\in I}|\langle f,f_i\rangle|^2 \le ||f||^2;$$

keeping only the terms corresponding to $i=1,2$ shows that

$$|\langle f,f_1\rangle|^2+|\langle f,f_2\rangle|^2 \le ||f||^2. \tag{4.3}$$

Putting (4.1) and (4.2) into this yields

$$\begin{aligned}&(||f_1||^4+|\langle f_1,f_2\rangle|^2)x^2+2\langle f_1,f_2\rangle(||f_1||^2+||f_2||^2)x+||f_2||^4+|\langle f_1,f_2\rangle|^2\\ &\le x^2||f_1||^2+2x\langle f_1,f_2\rangle+||f_2||^2,\end{aligned}$$

or,

$$\begin{aligned}&(||f_1||^2-||f_1||^4-|\langle f_1,f_2\rangle|^2)x^2+2\langle f_1,f_2\rangle(1-||f_1||^2-||f_2||^2)x\\ &+||f_2||^2-||f_2||^4-|\langle f_1,f_2\rangle|^2\ge 0.\end{aligned} \tag{4.4}$$

We split into two cases:
(1): Assume $||f_1||^2-||f_1||^4-|\langle f_1,f_2\rangle|^2=0$, or,

$$|\langle f_1,f_2\rangle|^2=||f_1||^2-||f_1||^4. \tag{4.5}$$

Note that (4.4) is satisfied for all real values of x. Thus,

$$\langle f_1,f_2\rangle(1-||f_1||^2-||f_2||^2)=0.$$

If $\langle f_1,f_2\rangle=0$, then (3.2) trivially holds; if $1-||f_1||^2-||f_2||^2=0$, then (4.5) implies that

$$\begin{aligned}|\langle f_1,f_2\rangle|^2 &= ||f_1||^2-||f_1||^4\\ &= (1-||f_1||^2)||f_1||^2\\ &= (1-||f_1||^2)(1-||f_2||^2),\end{aligned}$$

so (3.2) holds.

(2): Assume that $||f_1||^2-||f_1||^4-|\langle f_1,f_2\rangle|^2\neq 0$. Let

$$\begin{aligned}&a:=||f_1||^2-||f_1||^4-|\langle f_1,f_2\rangle|^2\ (\neq 0)\\ &b:=\langle f_1,f_2\rangle(1-||f_1||^2-||f_2||^2)\\ &c:=||f_2||^2-||f_2||^4-|\langle f_1,f_2\rangle|^2.\end{aligned} \tag{4.6}$$

Then (4.4) implies that

$$ax^2 + 2bx + c \geq 0.$$

Substitute $x := -b/a$ into this, to obtain

$$-(b^2 - ac)/a \geq 0. \tag{4.7}$$

The frame condition (4.3) applied to $f := f_1$ yields that

$$|\langle f_1, f_2 \rangle|^2 \leq ||f_1||^2 - ||f_1||^4,$$

so $a > 0$. It follows that

$$b^2 - ac \leq 0 \tag{4.8}$$

Using (4.6), a direct calculation shows that

$$\begin{aligned} b^2 - ac &= \left(|\langle f_1, f_2\rangle|^2 - ||f_1||^2||f_2||^2\right) \times \\ & \left(|\langle f_1, f_2\rangle|^2 - (1 - ||f_1||^2 - ||f_2||^2 + ||f_1||^2||f_2||^2)\right). \end{aligned}$$

By Cauchy-Schwarz inequality,

$$|\langle f_1, f_2\rangle|^2 \leq ||f_1||^2||f_2||^2.$$

This and (4.8) imply

$$|\langle f_1, f_2\rangle|^2 \leq 1 - ||f_1||^2 - ||f_2||^2 + ||f_1||^2||f_2||^2.$$

Thus (3.2) holds.

Now, we assume $\langle f_1, f_2\rangle$ is complex. Choose $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\lambda\langle f_1, f_2\rangle = |\langle f_1, f_2\rangle|$. Let $\tilde{f} := x\lambda f_1 + f_2$ for $x \in \mathbb{R}$. Then

$$||\tilde{f}||^2 = x^2||f_1||^2 + 2x|\langle f_1, f_2\rangle| + ||f_2||^2$$

and

$$\begin{aligned} |\langle \tilde{f}, f_1\rangle|^2 + |\langle \tilde{f}, f_2\rangle|^2 &= (||f_1||^4 + |\langle f_1, f_2\rangle|^2)x^2 + 2|\langle f_1, f_2\rangle|(||f_1||^2 + ||f_2||^2)x \\ &+ \ ||f_2||^4 + |\langle f_1, f_2\rangle|^2. \end{aligned}$$

Hence we can apply the partial result just proved to $\tilde{f}$.

□

Note that the correct value of the Bessel bound is essential in (3.2) :

Example 4.1 Let $\{e_1, e_2\}$ be an orthonormal basis for a 2-dimensional Hilbert space and put $f_1 = \sqrt{1+\epsilon}\, e_1, f_2 = \sqrt{1-\epsilon}\, e_2$ for some $\epsilon \in]0,1[$. Then $\{f_1, f_2\}$ is a Bessel sequence with bound $1+\epsilon$, and

$$\begin{aligned} 1 - ||f_1||^2 - ||f_1||^2 + ||f_1||^2||f_2||^2 &= 1 - (1+\epsilon) - (1-\epsilon) + (1+\epsilon)(1-\epsilon) \\ &= -\epsilon^2 < 0. \end{aligned}$$

By Lemma 3.1 the inequality (3.2) holds with $B = 1+\epsilon$. The above calculation shows that the inequality is false if B is replaced by 1. □

Acknowledgment: The authors thank the reviewers for useful comments that improved the presentation of the paper. In particular one reviewer suggested to use Example 2.7 instead of our original more complicated example.

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