Quasi-Interpolatory Refinable Functions and Construction of Biorthogonal Wavelet Systems

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Abstract

We present a new family of compactly supported and symmetric biorthogonal wavelet systems. Each refinement mask in this family has tension parameter ω. When $\omega = 0$, it becomes the minimal length biorthogonal Coifman wavelet system [17]. Choosing ω away from zero, we can get better smoothness of the refinable functions at the expense of slightly larger support. Though the construction of the new biorthogonal wavelet systems, in fact, starts from a new class of quasi-interpolatory subdivision schemes, we find that the refinement masks accidently coincide with the ones by Cohen, Daubechies and Feauveau [5, §6.C] (or [7, §8.3.5]), which are designed for the purpose of generating biorthogonal wavelets close to orthonormal cases. However, the corresponding mathematical analysis is yet to be provided. In this study, we highlight the connection between the quasi-interpolatory subdivision schemes and the masks by Cohen, Daubechies and Feauveau, and then we study the fundamental properties of the new biorthogonal wavelet systems such as regularity, stability, linear independence, vanishing moments and accuracy.

Keywords: Subdivision, Coifman Wavelet, Biorthogonal Wavelet, Multiresolution Analysis, Quasi-Interpolation, Refinable Function, Regularity, Linear Independence.

1 Introduction

During the last decades, the theory of wavelets and multiresolution analysis has established itself firmly as one of the most successful methods for a broad range of signal processing applications. The construction of classical wavelets is now well-understood due to pioneer works such as [5, 6, 7]. Many properties, such as symmetry (or antisymmetry), vanishing moments, regularity and short support, are required for a practical use in application areas. It has been well-known that orthogonality and symmetry are conflicting properties for the design of compactly supported wavelets [7]. In order to maintain the symmetric properties of wavelet systems, the orthogonality constraint has been relaxed to semi-orthogonality or biorthogonality. In particular, spline functions have been

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a good source for wavelet constructions. We select some of them from references [5, 2, 3]. Also, recently, a new class of compactly supported biorthogonal wavelet systems has been constructed from pseudo-splines in [9].

It is very common to introduce wavelets through the notion of multiresolution analysis [16] which is introduced as follows. First, we say that a function $\phi \in L_2(\mathbb{R})$ is a refinable function if it satisfies the so-called *refinement equation*

$$\phi(x) = \sum_{n\in\mathbb{Z}} a_n \phi(2x-n) \tag{1.1}$$

where $\mathbf{a} := \{a_n : n \in \mathbb{Z}\}$ is usually called *the refinement mask* for ϕ. The function ϕ is also termed as the basic limit function of a subdivision scheme with the mask $\mathbf{a}$ (see Definition 2.1). Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function and let V_j be a shift invariant space defined by

$$V_j = \overline{\operatorname{span}}\{\phi_{j,k} := 2^{j/2}\phi(2^j \cdot -k) : k \in \mathbb{Z}\}.$$

We say that a sequence of subspaces $\{V_j : j \in \mathbb{Z}\}$ forms a *multiresolution analysis* (MRA) if it satisfies the following conditions:

(1) $\{V_j : j \in \mathbb{Z}\}$ is nested, i.e., $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.
(2) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(3) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$.
(4) The set of the translates $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is a *Riesz basis* for the space V_0; see Section 3.

Let $\{V_j : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j : j \in \mathbb{Z}\}$ be a pair of MRAs. The concept of biorthogonal wavelets consists of finding complement space W_j and $\tilde{W}_j$ of V_j and $\tilde{V}_j$ respectively satisfying

$$\tilde{W}_j \perp V_j, \quad W_j \perp \tilde{V}_j$$

so that $W_j \perp \tilde{W}_\ell$ for $j \neq \ell$.

Our construction of biorthogonal wavelet systems starts from a new class of subdivision schemes [1] (say, S_L). The reader is referred to the paper [1] to find its interesting features in view of CAGD (computer aided geometric design). Each scheme in this class is a quasi-interpolatory scheme, which reproduces polynomials up to a certain degree, with a tension parameter ω. When $\omega = 0$, it becomes the Deslauriers-Dubuc's interpolatory scheme, whose mask is used as a key ingredient to construct the (minimal length) biorthogonal Coifman wavelets. A biorthogonal wavelet system with compact support is called a biorthogonal Coifman wavelet system for degree L if the synthesis refinable function $\phi(x)$ and the dual wavelets $\psi(x)$ and $\tilde{\psi}(x)$ have the vanishing moments L, that is,

$$\begin{aligned} &\int_{\mathbb{R}} x^n \phi(x)dx = \delta_{0,n}, \quad \forall n = 0, \dots, L, \\ &\int_{\mathbb{R}} x^n \psi(x)dx = \int_{\mathbb{R}} x^n \tilde{\psi}(x)dx = 0, \quad \forall n = 0, \dots, L. \end{aligned} \tag{1.2}$$

Thus, the main objective of this paper is to present and to analyze a new family of biorthogonal Coifman wavelet systems which are symmetric and compactly supported. An interesting observation is that the refinement masks of the new family coincide with the ones by Cohen, Daubechies and Feauveau [5, §6.C] (see also [7, §8.3.5]), which are designed for the purpose of generating biorthogonal filters close to orthonormal cases (in fact, to some coiflets). However, the corresponding mathematical analysis has not been provided yet. In this study, we highlight the connection between the quasi-interpolatory subdivision schemes S_L and the refinement masks in [5, 7]. Furthermore, we analyze the mathematical properties associated with the refinable functions and wavelets such as stability, linear independence, regularity, and vanishing moments. We then will enjoy the following advantages of the suggested wavelet systems:

- Choosing ω away from zero, the corresponding refinable functions have better smoothness, at the expense of slightly larger support, than the ones of $\omega = 0$. For instance, the suggested refinable function based on the cubic polynomial can be $H^{3.7074}$ in the sense of Hölder regularity, while the one by the cubic polynomial-based Deslauriers-Dubuc scheme is $H^{2-\epsilon}$ and the cubic B-spline refinable function is $H^{3-\epsilon}$.

- One attractive property of the new wavelet systems is that some filter coefficients can be dyadic rationals, i.e., rationals of the form $(2p+1)/2^q$ for some positive integers p and $q > 0$; since division by 2 can be done very fast in a computer, this makes it very suitable for fast computation.

- The coefficients in the biorthogonal projection $P_j f$ onto the space V_j (see (5.8)) can be replaced by the sampled values of a function f, keeping the optimal convergence order of the error $\|f - P_j f\|_{L_2(\mathbb{R})}$. This amounts to avoiding the calculation of the inner products of approximands f and refinable functions.

- For some suitable values of ω, the corresponding biorthogonal wavelet systems are very close to orthonormal cases. For an algorithm to find ω and details, the reader should consult [5].

The article is organized in the following manner: In Section 2, we briefly introduce the quasi-interpolatory subdivision schemes along with the Deslauriers-Dubuc interpolatory scheme. Some analysis on their masks is also given. In Section 3, we find the condition of ω which guarantees the linear independence of the integer translates of the refinable function ϕ associated with the quasi-interpolatory scheme. The regularity of the refinable functions are studied in Section 4. In Section 5, we construct a new class of biorthogonal wavelet systems, which are symmetric and compactly supported. Finally, we show some specific examples of biorthogonal wavelet systems based on cubic polynomial.

2 Quasi-Interpolatory Subdivision Schemes

2.1 Subdivision Scheme

Starting with the initial values $f^0 = \{f_n^0 \in \mathbb{R} : n \in \mathbb{Z}\}$, a subdivision scheme defines recursively new discrete values $f^j = \{f_n^j \in \mathbb{R} : n \in \mathbb{Z}\}$ on finer levels by linear sums of existing values as follows:

$$f_\ell^{j+1} = \sum_{n\in\mathbb{Z}} a_{\ell-2n} f_n^j, \quad j \in \mathbb{Z}_+, \tag{2.1}$$

where the sequence $\mathbf{a} = \{a_n : n \in \mathbb{Z}\}$ is termed the *mask* of the given subdivision. We denote the rule at each level by S and have the formal relation

$$f^j = S^j f^0. \tag{2.2}$$

Definition 2.1 *A subdivision scheme S is said to be C^ν if for the initial data $f^0 := \{\delta_{n,0} : n \in \mathbb{Z}\}$, there exists a limit function $\phi := S^\infty f^0 \in C^\nu(\mathbb{R})$, $\phi \not\equiv 0$, satisfying*

$$\lim_{j\to\infty} \sup_{n\in\mathbb{Z}} |f_n^j - \phi(2^{-j}n)| = 0. \tag{2.3}$$

The function ϕ is called the *basic limit function* of S and it satisfies the *refinement equation* in (1.1) [11]. It is obvious that supp $\phi \subseteq$ [supp $\mathbf{a}$] where $[A]$ indicates the smallest closed interval containing the set A.

To simplify the presentation of a subdivision scheme and its analysis, it is convenient to introduce the Laurent polynomial defined by $\mathbf{a} = \{a_n : n \in \mathbb{Z}\}$

$$a(z) := \sum_{n\in\mathbb{Z}} a_n z^n, \quad z \in \mathbb{C} \setminus \{0\}. \tag{2.4}$$

The Laurent polynomial $a(z)$ is also called the symbol of its corresponding refinable function ϕ; see (2.3). Next, define the Laurent polynomial $a^{[j]}(z)$, $j \in \mathbb{N}$, by

$$a^{[j]}(z) := \sum_{n\in\mathbb{Z}} a_n^{[j]} z^n := a(z)a(z^2)\cdots a(z^{2^{j-1}}). \tag{2.5}$$

Using the coefficients $a_n^{[j]}$ in (2.5), the norm of the iterated scheme S^j in (2.2) is defined as the following [11]:

$$\|S^j\|_\infty := \max\Big\{ \sum_{\beta\in\mathbb{Z}} |a^{[j]}_{\gamma+2^j\beta}| : \gamma = 0, \ldots, 2^j - 1 \Big\}. \tag{2.6}$$

2.2 The Mask of Quasi-Interpolatory Subdivision Scheme

As observed in (2.1), a univariate subdivision consists of two rules, which can be represented by the even and the odd masks. First, for the construction of the odd mask, we use the stencil of $L = 2N$ points to reproduce polynomials of degree $< 2N$. That is, the odd mask $\{a_{1-2n} : n = -N+1, \ldots, N\}$ is obtained by solving the linear system:

$$p_\ell(2^{-1}) = \sum_{n=-N+1}^{N} a_{1-2n} p_\ell(n), \quad \ell = 1, \ldots, L, \tag{2.7}$$

where p_ℓ, $\ell = 1, \ldots, L$, is a basis of $\Pi_{<L}$. Obviously, there is a unique solution of the linear system (2.7) and it is exactly the same as the odd mask of the L-point Deslauriers-Dubuc scheme, i.e.,

$$\begin{aligned} a_{1-2n} &= L_n(1/2) \\ &= \frac{(-1)^n}{1-2n} \binom{2N-2}{N-1} \binom{2N-1}{N-n} \frac{2N-1}{2^{4N-3}}. \end{aligned}$$

Here, $L_n(x)$ is the Lagrange polynomials on $\{-N+1, \ldots, N\}$ defined by

$$L_n(x) = \prod_{\substack{\ell \neq n \\ \ell=-N+1}}^{N} \frac{x-\ell}{n-\ell}, \quad n = -N+1, \ldots, N, \tag{2.8}$$

so that

$$p_\ell(x) = \sum_{n=-N+1}^{N} L_n(x) p_\ell(n) \tag{2.9}$$

where $p_1, \ldots, p_L$ constitute a basis of $\Pi_{<L}$.

Next, for the construction of the even mask, we use the stencil of $L+1 = 2N+1$ points to reproduce polynomials in $\Pi_{<L}$. That is, the even mask $\{a_{2n} : n = -N, \ldots, N\}$ is obtained by solving the linear system:

$$p_\ell(0) = \sum_{n=-N}^{N} a_{-2n} p_\ell(n), \quad p_\ell \in \Pi_{<L}. \tag{2.10}$$

This is an underdetermined system of $L+1$ unknowns a_{2n}, $n = -N, \ldots, N$, in L equations, and hence there is one degree of freedom which will be used as a tension parameter ω. Here and in the sequel, for convenience, we put

$$\omega := a_{2N}.$$

The following lemma treats the explicit formula of the even mask a_{2n}.

Lemma 2.2 *Let $\{a_{2n} : n = -N, \ldots, N\}$ be the even mask of the subdivision scheme S_L obtained from* (2.10) *and let $a_{2N} = \omega$. Then, for $n = -N+1, \ldots, N$,*

$$a_{-2n} = \delta_{n,0} - \omega b_{-2n}. \tag{2.11}$$

where

$$b_{-2n} = (-1)^{N-n+1} \frac{(2N)!}{(N+n)!(N-n)!}. \tag{2.12}$$

Proof. Using $a_{2N} = \omega$, the linear system (2.10) can be changed to

$$\sum_{n=-N+1}^{N} a_{-2n} p_\ell(n) = p_\ell(0) - \omega p_\ell(-N), \quad p_\ell \in \Pi_{<L}. \tag{2.13}$$

This is a $2N \times 2N$ system and it guarantees the unique solution of (2.13). Let $\mathbf{M}$ and $\mathbf{R}_x$ be matrices defined by $\mathbf{M}(\ell, k) = p_\ell(k-N)$, $k, \ell = 1, \ldots, L$, and $\mathbf{R}_x(\ell) = p_\ell(x)$, $\ell = 1, \ldots L$. The solution $\mathbf{A}_{\text{even}} = \{a_{-2n} : n = -N+1, \ldots, N\}$ can be expressed in the matrix form

$$\mathbf{A}_{\text{even}} = \mathbf{M}^{-1}(\mathbf{R}_0 - \omega \mathbf{R}_{-N}).$$

From (2.9), it is obvious that

$$\mathbf{M}^{-1}\mathbf{R}_0 = (L_n(0) : n = -N+1, \ldots, N),$$
$$\mathbf{M}^{-1}\mathbf{R}_{-N} = (L_n(-N) : n = -N+1, \ldots, N),$$

with $L_n(x)$ the Lagrange polynomial in (2.8). Here, for any $n = -N+1, \dots, N$, we have

$$
\begin{aligned}
L_n(-N) &= \prod_{\substack{\ell \neq n \\ \ell=-N+1}}^{N} \frac{-N-\ell}{n-\ell} \\
&= \frac{(-1)^{2N}(2N)!}{-N-n} \cdot \frac{(-1)^{N-n}}{(N-n)!} \cdot \frac{1}{(N+n-1)!} \\
&= (-1)^{N-n+1} \frac{(2N)!}{(N+n)!(N-n)!}.
\end{aligned}
$$

Denoting the last term by b_{-2n}, the proof is done. □

Remark 2.3 *From Lemma 2.2, it is obvious that if $\omega = 0$, $a_{2n} = \delta_{n,0}$. This implies that the $2N$-point Deslauriers-Dubuc scheme is a special case of S_L with $\omega = 0$ and $L = 2N$.*

The next lemma provides the explicit form of the Laurent polynomial $a(z)$ associated with the scheme S_L.

Lemma 2.4 *Let $\{a_n : n \in \mathbb{Z}\}$ be the mask of the subdivision scheme S_L with $L = 2N$ and $a(z)$ be its corresponding Laurent polynomial. If we set $a_{2N} = \omega$ and $y = \sin^2(\xi/2)$, then*

$$
a(e^{i\xi}) = (1-y)^N \left[2 \sum_{n=0}^{N-1} \binom{N-1+n}{n} y^n + \omega 2^{4N} (-1)^N y^N \right]. \tag{2.14}
$$

Proof. The Laurent polynomial of the $2N$-point Deslauriers-Dubuc scheme can be written as

$$
a_D(z) := 1 + \sum_{n=-N+1}^{N} a_{1-2n} z^{1-2n}. \tag{2.15}
$$

Using this expression and applying Lemma 2.2, we get

$$
\begin{aligned}
a(z) &= \omega z^{2N} + \sum_{n=-N+1}^{N} \Big((\delta_{n,0} - \omega b_{-2n}) z^{-2n} + a_{1-2n} z^{1-2n} \Big) \\
&= \omega \Big(z^{2N} - \sum_{n=-N+1}^{N} b_{-2n} z^{-2n} \Big) + a_D(z)
\end{aligned} \tag{2.16}
$$

with b_{-2n} as in (2.12). It is well known from the literature (e.g., see [7]) that the Laurent polynomial $a_D(z)$, $z = e^{i\xi}$, has the explicit form

$$
a_D(e^{i\xi}) \;=\; 2\cos^{2N}(\xi/2) \sum_{n=0}^{N-1} \binom{N-1+n}{n} \sin^{2n}(\xi/2). \tag{2.17}
$$

Moreover, invoking the definition of b_{-2n} in (2.12) and using $z = e^{i\xi}$, we have

$$
\begin{aligned}
z^{2N} - \sum_{n=-N+1}^{N} b_{-2n} z^{-2n} &= z^{2N} - \sum_{n=-N+1}^{N} (-1)^{N-n+1} \frac{(2N)!}{(N+n)!(N-n)!} z^{-2n} \\
&= z^{2N} \left(1 + \sum_{n=1}^{2N} (-1)^{-n} \frac{(2N)!}{n!(2N-n)!} z^{-2n} \right) \\
&= z^{2N}(1-z^{-2})^{2N} \\
&= z^{-2N} 2^{4N} i^{2N} \left(\frac{1+z}{2} \right)^{2N} \left(\frac{1-z}{2i} \right)^{2N} \\
&= 2^{4N} i^{2N} \cos^{2N}(\xi/2) \sin^{2N}(\xi/2). \qquad (2.18)
\end{aligned}
$$

Combining (2.17) and (2.18) with (2.16),

$$
\begin{aligned}
a(e^{i\xi}) &= \cos^{2N}(\xi/2) \left[2 \sum_{n=0}^{N-1} \binom{N-1+n}{n} \sin^{2n}(\xi/2) + \omega 2^{4N} i^{2N} \sin^{2N}(\xi/2) \right] \\
&= (1-y)^N \left[2 \sum_{n=0}^{N-1} \binom{N-1+n}{n} y^n + \omega 2^{4N} (-1)^N y^N \right]
\end{aligned}
$$

where $y = \sin^2(\xi/2)$. This completes the proof. $\square$

We can find that the Laurent polynomial $a(z)$ in (2.14) coincides with the one in [5, §6.C] (see also [7, §8.3.5]), which was designed for the purpose of constructing biorthogonal filters close to orthonormal cases (or to some coiflets). However, the mathematical properties of the corresponding refinable functions are not studied. In the following sections, we will discuss the fundamental properties of $a(z)$ in (2.14) in relation to the linear independence and the smoothness of the corresponding refinable functions.

3 Linear Independence and Stability of Refinable Functions

Given a refinable function ϕ, a fundamental question is whether its integer translates are linearly independent: The integer translates of a compactly supported function $\phi \in L_2(\mathbb{R})$ are linearly independent if for any $c \in \ell(\mathbb{Z})$,

$$
\sum_{j \in \mathbb{Z}} c(j)\phi(\cdot - j) = 0 \quad \text{implies} \quad c(j) = 0, \quad \forall j \in \mathbb{Z}.
$$

The linear independence of the integer translates of ϕ is a necessary and sufficient condition for the existence of a compactly supported dual refinable function $\tilde{\phi} \in L_2(\mathbb{R})$ of ϕ (see [15]). Furthermore, it is well-known that the existence of a compactly supported dual refinable function of ϕ is a key step to construct a pair of biorthogonal wavelets from the given ϕ.

An issue related to the linear independence is the (somewhat weaker) notion of the stability of ϕ: A function $\phi \in L_2(\mathbb{R})$ is *stable* if there exist $0 < A, B < \infty$ such that for any sequence $c \in \ell_2(\mathbb{Z})$,

$$
A\|c\|_{\ell_2(\mathbb{Z})} \le \Big\| \sum_{j \in \mathbb{Z}} c(j)\phi(\cdot - j) \Big\|_{L_2(\mathbb{R})} \le B\|c\|_{\ell_2(\mathbb{Z})}. \qquad (3.1)
$$

In other words, the integer translates of ϕ are stable if the collection $\{\phi(\cdot - j) : j \in \mathbb{Z}\}$ is an *unconditional basis* for the subspace of $L_2(\mathbb{R})$ generated by them. The upper bound of (3.1) always exists for any compactly supported function $\phi \in L_2(\mathbb{R})$ [12, Theorem 2.1]. It is also well-known from [12, Theorem 3.5] that the lower bound is equivalent to

$$\big(\hat{\phi}(\xi + 2\pi k)\big)_{k\in\mathbb{Z}} \neq \mathbf{0}, \quad \forall \xi \in \mathbb{R}, \tag{3.2}$$

where $\mathbf{0}$ indicates the zero sequence in $\ell(\mathbb{Z})$. Thus, the stability of a compactly supported function $\phi \in L_2(\mathbb{R})$ is equivalent to (3.2).

The linear independence of the integer translates of a refinable function ϕ is characterized in terms of their masks in [13]. The main results, Theorem 1 and 2 in [13], imply directly the following lemma. Here, the notion of symmetric zeros is used: A Laurent polynomial $a(z)$ has a pair of symmetric zeros on $\mathbb{C} \setminus \{0\}$ if there is a zero $z_0 \in \mathbb{C} \setminus \{0\}$ such that $a(z_0) = a(-z_0) = 0$.

Lemma 3.1 [13] *Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function. The integer translates of ϕ are linearly independent if and only if the following two conditions are satisfied:*

(1) *The function ϕ is stable.*
(2) *The Laurent polynomial $a(z)$ does not have any symmetric zeros in $\mathbb{C} \setminus \{0\}$.*

From the above lemma, it is apparent that for a compactly supported function $\phi \in L_2(\mathbb{R})$, the linear independence of the integer translates of ϕ implies the stability of ϕ. In what follows, we find the condition on ω which guarantees the linear independence of the integer translates of ϕ associated with the subdivision scheme S_L. For this, two useful lemmas are introduced.

Lemma 3.2 *Let*

$$P_N(y) := \sum_{n=0}^{N-1} \binom{N-1+n}{n} y^n \tag{3.3}$$

and

$$Q_N(y) := P_N(y) - (-1)^{N+1} 2^{4N-1} \omega y^N. \tag{3.4}$$

Then $Q_N(y)$ and $Q_N(1-y)$ do not vanish simultaneously for $y \in [0,1]$ if and only if

$$\omega \neq (-1)^{N+1} 2^{-2N}.$$

Proof. First we claim that the polynomial Q_N has at most one zero for $y \in [0,1]$. For this proof, suppose that $Q_N(y_1) = 0$. Clearly, $y_1 \neq 0$, and it follows from (3.4) that

$$P_N(y_1) = (-1)^{N+1} 2^{4N-1} \omega y_1^N. \tag{3.5}$$

Then, for any $y \neq y_1$, we apply (3.4) to get the relation

$$\begin{aligned} Q_N(y) &= P_N(y) - P_N(y_1) (\frac{y}{y_1})^N \\ &= \sum_{j=0}^{N-1} \binom{N-1+j}{j} y^j \left(1 - (\frac{y}{y_1})^{N-j}\right), \end{aligned}$$

where $Q_N(y) > 0$ if $y < y_1$ and $Q_N(y) < 0$ if $y > y_1$; hence Q_N has at most one zero in $[0, 1]$. Therefore, $Q_N(y)$ and $Q_N(1-y)$ do not vanish simultaneously if and only if $Q_N(1/2) \neq 0$, that is, by (3.4), equivalent to

$$\omega \neq (-1)^{N+1} 2^{-3N+1} P_N(1/2). \tag{3.6}$$

Finally, we need to evaluate $P_N(1/2)$. Since the polynomial P_N satisfies the equation [7]

$$y^N P_N(1-y) + (1-y)^N P_N(y) = 1,$$

we get $P_N(1/2) = 2^{N-1}$ by setting $y = 1/2$. Combining this with (3.6) leads to the claim of the lemma. $\square$

Remark 3.3 *According to the condition* (3.6) *and its subsequent argument, we conclude that if* $\omega = (-1)^{N+1} 2^{-2N}$*, the polynomial* $Q_N(y)$ *has one root at* $y = 1/2$*, i.e.,* $Q_N(1/2) = 0$.

Lemma 3.4 [13, Theorem 1] *Let* $\phi \in L_2(\mathbb{R})$ *be a compactly supported refinable function with the symbol* $a(z)$*. Then* ϕ *is stable if and only if the following two conditions are satisfied:*

(1) *The symbol* $a(z)$ *does not have any symmetric zeros on the unit circle* T*.*
(2) *For any odd integer* $m > 1$ *and a primitive* m*-th root* z_0 *of unity, there is* $d \in \mathbb{N}$ *such that*

$$a(-z_0^{2^d}) \neq 0.$$

Based on this lemma, we prove that the refinable function ϕ associated with S_L is stable. Invoking the definition of Q_N in (3.4), it is useful for the following analysis to represent $a(e^{i\xi})$ in (2.14) as

$$a(e^{i\xi}) = 2(1-y)^N Q_N(y), \quad y = \sin^2(\xi/2). \tag{3.7}$$

Lemma 3.5 *Let* $\phi \in L_2(\mathbb{R})$ *be the refinable function associated with the subdivision scheme* S_L *with the tension parameter* ω*. Then* ϕ *is stable if and only if*

$$\omega \notin (-1)^{N+1} 2^{-2N+1} \{1/2, P_N(1/4)\}.$$

Proof. We prove this lemma by using Lemma 3.4. Due to Lemma 3.2, it suffices to show that the condition (2) in Lemma 3.4 is equivalent to the condition

$$\omega \neq (-1)^{N+1} 2^{-2N+1} P_N(1/4).$$

Let z_0 be a primitive 3-rd root of unity, *i.e.*, $z_0 = e^{2\pi i/3}$ or $e^{4\pi i/3}$. We first claim that $a(-z_0^{2^d}) = 0$ for all $d \geq 0$ if and only if $\omega = (-1)^{N+1} 2^{-2N+1} P_N(1/4)$. For this proof, define ζ_0 by $z_0 = e^{i\zeta_0}$. A direct calculation shows that for any integer $d \geq 0$, $\sin^2(2^{d-1}\zeta_0) = 3/4$. Therefore, we obtain from (3.4) that for all $d \geq 0$,

$$\begin{aligned} a(-z_0^{2^d}) &= 2\sin^{2N}(2^{d-1}\zeta_0) Q_N(\cos^2(2^{d-1}\zeta_0)) \\ &= 2\sin^{2N}(2^{d-1}\zeta_0)(P_N(1/4) - (-1)^{N+1} 2^{4N-1} \omega 4^{-N}), \end{aligned} \tag{3.8}$$

such that $a(-z_0^{2^d}) = 0$ if and only if $\omega = (-1)^{N+1}2^{-2N+1}P_N(1/4)$. Now, assume that $\omega \neq (-1)^{N+1}2^{-2N+1}P_N(1/4)$ and that there exists $z_1 = e^{i\zeta_1} \in \mathbb{C} \setminus \{0, z_0\}$ with $\zeta_1 \in (0, 2\pi)$ such that $a(-e^{i2^d\zeta_1}) = 0$ for all integer $d \geq 0$. Let $y_1 := \sin^2(\zeta_1/2)$. Then, let us consider the cases $d = 0$ and 1. If $d = 0$,

$$a(-e^{\zeta_1 i}) = 2y_1^N Q_N(1 - y_1) = 0$$

and if $d = 1$,

$$a(-e^{2\zeta_1 i}) = 2(4y_1(1 - y_1))^N Q_N((1 - 2y_1)^2) = 0.$$

It follows that $Q_N(1 - y_1) = Q_N((1 - 2y_1)^2) = 0$. Since $Q_N(y)$ has at most one zero (as observed in the proof of Lemma 3.2), $1 - y_1$ should be equal to $(1 - 2y_1)^2$. It implies that $y_1 = 3/4$. But, due to (3.8), $Q_N(1 - y_1) = Q_N(1/4) \neq 0$ since $\omega \neq (-1)^{N+1}2^{-2N+1}P_N(1/4)$. This proves the lemma. □

Lemma 3.6 *Let $a(z)$ be the Laurent polynomial of the subdivision scheme S_L and let b_{2n} for $n = -N + 1, \ldots, N$ be given as in (2.12). Assume that $z_0 \in \mathbb{C} \setminus \{0\}$ be a zero of $a(z)$. Then, z_0 is a symmetric zero of $a(z)$ if and only if*

$$\omega = \left[\sum_{n=-N+1}^{N} b_{-2n} z_0^{-2n} - z_0^{2N} \right]^{-1}. \tag{3.9}$$

Proof. Recalling the expression of $a(z)$ in (2.16), let $z_0 \in \mathbb{C} \setminus \{0\}$ be a symmetric zero of $a(z)$. Dividing $a(z)$ into even and odd degree terms and using the condition $a(z_0) = a(-z_0) = 0$, it can be easily induced that

$$a_{\mathrm{odd}}(z_0) := \sum_{n \in \mathbb{Z}} a_{1-2n} z_0^{1-2n} = 0. \tag{3.10}$$

Since the mask $\{a_{1-2n} : n \in \mathbb{Z}\}$ is exactly the same as the case of the L-point Deslauriers-Dubuc scheme, it is clear that $a_D(z_0) = 1$ with $a_D(z)$ in (2.15). Thus, by (2.16), we obtain the required condition (3.9). □

Theorem 3.7 *Let $\phi \in L_2(\mathbb{R})$ be the refinable function generated by the subdivision scheme S_L, $L = 2N$, with a tension parameter ω. Assume that*

$$\omega \notin (-1)^{N+1}2^{-2N+1}\big\{1/2, P_N(1/4)\big\} \cup \mathcal{V}_N \tag{3.11}$$

where

$$\mathcal{V}_N = \left\{ \Big(\sum_{n=-N+1}^{N} b_{-2n} z_0^{-2n} - z_0^{2N} \Big)^{-1} : a_{\mathrm{odd}}(z_0) = 0,\ z_0 \in \mathbb{C} \setminus \{0\} \right\}$$

Then the integer translates of ϕ are linearly independent.

Proof. We check the two sufficient conditions in Lemma 3.1. The condition (1) is proved in Lemma 3.5. The condition (2) is also an immediate consequence of Lemma 3.6. □

L	γ	ω	L	γ	ω
2	2.9999	$\frac{1}{8}$	12	6.6976	$-.000078$
4	3.7074	$-.024397$	14	7.4445	.000019
6	4.6783	.005632	16	8.1887	$-.00000462$
8	5.4800	$-.001348$	18	8.5983	.00000109
10	6.1221	.000332	20	9.4260	$-.00000027$

Table 1: The maximum Hölder regularities H^γ of ϕ associated with S_L for each $L = 2, \ldots, 20$ and the corresponding values of ω. These are computed by using MAPLE 8, digits = 15.

4 Smoothness Analysis

4.1 Maximal Smoothness of Refinable Functions

Let ϕ be the refinable function associated with the subdivision scheme S_L. An interesting observation is that as the tension parameter ω is away from zero (up to a suitable range), the smoothness of ϕ is increased. Here, we discuss the maximal Hölder smoothness of ϕ for each given L. For a given $\gamma = n + s$ with $n \in \mathbb{N}$ and $s \in [0, 1)$, the Hölder space H^γ is defined as the space of n-times continuously differentiable functions f whose n-th derivative $f^{(n)}$ satisfies the Lipschitz condition

$$\sup_{x,h\in\mathbb{R}} \frac{|f^{(n)}(x+h) - f^{(n)}(x)|}{|h|^s} \le C.$$

In particular, it is well known (e.g., see Lemma 7.1 of [8]) that if $|\hat{f}(\xi)| \le c(1 + |\xi|)^{-1-\gamma-\epsilon}$ with $\epsilon > 0$, then f belongs to the space H^γ.

The specific maximal Hölder regularity of ϕ generated by S_L is obtained in Table 1 with the corresponding values of ω. For this computation, we used Corollary 3.3 and Theorem 3.4 in [11, section 2.3]. It is remarkable to see that the refinable function ϕ based on cubic polynomial (i.e., by S_L with $L = 4$) is $H^{3.7074}$, while the cubic-based Deslauriers-Dubuc's interpolatory scheme is $H^{2-\epsilon}$ and the cubic B-spline is $H^{3-\epsilon}$.

4.2 Asymptotic smoothness

For a suitable ω, we expect that the Hölder regularity of ϕ associated with S_L, $L = 2N$, increases as N is increasing. In what follows, we estimate its asymptotic property of a regularity of ϕ for the special choice of $\omega = \omega_N$ with

$$\omega_N := 2^{-4N+1}(-1)^{N+1}\frac{N-1}{N+1}\binom{2N-1}{N-1}, \tag{4.1}$$

as N tends to ∞. Of course, it does not provide the best smoothness among all possible ranges of ω for a fixed L, but at least we can compare it with the asymptotic property of the regularity of the case $\omega = 0$, as $N \to \infty$. With the choice of $\omega = \omega_N$, the Laurent polynomial $a(z)$ becomes of the form

$$a(e^{i\xi}) = 2\cos^{2N}(\xi/2)Q_N(\sin^2(\xi/2))$$

with

$$Q_N(y) := \left[\sum_{n=0}^{N-1} \binom{N-1+n}{n} y^n - \frac{N-1}{N+1} \binom{2N-1}{N-1} y^N \right], \quad y = \sin^2 \xi/2. \tag{4.2}$$

Theorem 4.1 *For a given $L = 2N$, let ω_N be given as in (4.1). Let ϕ be the refinable function associated with S_L and ω_N. Then, we have the optimal decay*

$$|\hat{\phi}(\xi)| \le C(1 + |\xi|)^{-2N+\kappa}, \tag{4.3}$$

where $\kappa = \log(|Q_N(3/4)|)/\log 2$ with Q_N in (4.2). Consequently, $\phi \in H^{2N-\kappa-1-\epsilon}$ for any $\epsilon > 0$.

We put the proof of Theorem 4.1 in Section 4.3 for the better readability of the article. The following theorem treats the special case $\omega = 0$, which is the case of the Deslauriers-Dubuc interpolatory scheme. Then, with P_N in (3.3), the Laurent polynomial $a_D(z)$ can be written as

$$a_D(e^{i\xi}) = 2\cos^{2N}(\xi/2) P_N(\sin^2(\xi/2)).$$

Theorem 4.2 [4, 7] *Let ϕ be the refinable function associated with the Deslauriers-Dubuc interpolatory scheme. Then, we have the optimal decay*

$$|\hat{\phi}(\xi)| \le C(1 + |\xi|)^{-2N+\tilde{\kappa}}, \tag{4.4}$$

where $\tilde{\kappa} = \log(|P_N(3/4)|)/\log 2$ with P_N in (3.3). Consequently, $\phi \in H^{2N-\tilde{\kappa}-1-\epsilon}$ for any $\epsilon > 0$.

Remark 4.3 *It is easy to check that $|P_N(3/4)| > |Q_N(3/4)|$; see also Lemmas 4.5 and 4.6. Hence,the suggested refinable function ϕ in Theorem 4.1 has better smoothness than the case of Deslauriers-Dubuc interpolatory scheme with the amount of*

$$\tilde{\kappa} - \kappa = \frac{1}{\log 2} \log \left(\frac{|P_N(3/4)|}{|Q_N(3/4)|} \right) > 0.$$

4.3 Proof of Theorem 4.1

We cite the following result from [10] (see also [7, Lemma 7.1.7]).

Proposition 4.4 *Let ϕ be the refinable function with the symbol $a(z)$ of the form*

$$|a(e^{i\xi})| := 2(1-y)^N |Q(y)|, \ y = \sin^2(\xi/2),$$

for some polynomial Q. Suppose that

(1) $|Q(y)| \le |Q(3/4)|$ *for* $0 \le y \le 3/4$*;*

(2) $|Q(y)Q(4y(1-y))| \le |Q(3/4)|^2$ *for* $3/4 \le y \le 1$*.*

Then, we have the optimal decay

$$|\hat{\phi}(\xi)| \le C(1 + |\xi|)^{-2N+\kappa},$$

with $\kappa = \log(|Q(3/4)|)/\log 2$. Consequently, $\phi \in C^{2N-\kappa-1-\epsilon}$ for any $\epsilon > 0$.

In order to prove Theorem 4.1, we will show that the symbol

$$a_N(e^{i\xi}) = 2(1-y)^N Q_N(y), \quad y = \sin^2(\xi/2)$$

satisfies the hypothesis of Proposition 4.4. To this end, we need the following lemmas.

Lemma 4.5 *Let $\ell \le N$ be a positive integer. The following statements hold:*

(1) *For any $j \in \mathbb{N}$, $(j+1)\binom{N+j}{j+1} = (N+j)\binom{N-1+j}{j}$.*

(2) $\displaystyle\sum_{j=0}^{\ell} \binom{N-1+j}{j} j = \frac{\ell(\ell+1)}{N+1}\binom{N+\ell}{\ell+1}$.

Proof. The relation (1) is trivial, and (2) can be shown by induction on ℓ. □

Lemma 4.6 *Let f be a polynomial of the form*

$$f(y) = \sum_{j=0}^{N-1} a_j y^j - b_N y^N,$$

with $a_j > 0$, $j = 0, \ldots, N-1$ and $b_N = \frac{1}{N}\sum_{j=1}^{N-1} j a_j$, so that $f'(1) = 0$. Then f is positive and increasing on $[0,1]$.

Proof. Note that $f(0) = a_0 > 0$. It follows from the choice of b_N that

$$f'(y) = \sum_{j=1}^{N-1} j a_j (1 - y^{N-j}) y^{j-1} \ge 0, \quad \forall y \in [0,1].$$

Therefore f is positive and increasing on $[0,1]$. □

For any $\ell = 0, 1, \ldots, N-1$, define a polynomial $Q_{N,\ell}$ by

$$Q_{N,\ell}(y) := \sum_{j=0}^{\ell} \binom{N-1+j}{j} y^j - \nu_\ell \, y^{\ell+1} \tag{4.5}$$

with

$$\nu_\ell := \frac{\ell}{N+1}\binom{N+\ell}{\ell+1} = \frac{1}{\ell+1}\sum_{j=0}^{\ell} j \binom{N-1+j}{j}. \tag{4.6}$$

Lemma 4.7 *For any $\ell = 0, \ldots, N-1$, $Q_{N,\ell}$ in (4.5) satisfies the following properties:*

(1) $\displaystyle Q_{N,\ell+1}(y) = Q_{N,\ell}(y) + \nu_{\ell+1} y^{\ell+1}\left(\frac{\ell+2}{\ell+1} - y\right).$ (4.7)

(2) $\displaystyle Q'_{N,\ell+1}(y) = \sum_{j=0}^{\ell} (N+j)\binom{N-1+j}{j}\left(y^j - y^{\ell+1}\right).$ (4.8)

Proof. From Lemma 4.5 (1), we rewrite $Q_{N,\ell}(y)$ as follows:

$$\begin{aligned} Q_{N,\ell}(y) &= \sum_{j=0}^{\ell} \binom{N-1+j}{j} \left(y^j - \frac{j}{\ell+1} y^{\ell+1} \right) \\ &= \sum_{j=0}^{\ell+1} \binom{N-1+j}{j} \left(y^j - \frac{j}{\ell+1} y^{\ell+1} \right) \\ &= \sum_{j=0}^{\ell+1} \binom{N-1+j}{j} y^j - \frac{\ell+2}{\ell+1} \nu_{\ell+1} y^{\ell+1}. \end{aligned} \tag{4.9}$$

Thus the relation (4.7) is immediate from the definition of $Q_{N,\ell+1}$ in (4.5). Also, using (4.6) and Lemma 4.5 (1), we get the relation in (2). □

We now proceed to the proof of Theorem 4.1. We follow the method in [9] but the presentation become simpler by introducing this function

$$\Lambda_j(y) := \Lambda_{\ell,j}(y) := y^j \left(1 - \frac{\ell+1}{\ell+2} y \right). \tag{4.10}$$

Proof of Theorem 4.1: We check the conditions (1) and (2) of Proposition 4.4. Using Lemma 4.5 and the identity $\binom{2N-1}{N-1} = \binom{2N-1}{N}$, it is easy to see that $Q_N(y)$ satisfies the hypothesis of Lemma 4.6 and hence, $Q_N(y)$ is positive and monotonically increasing on $[0,1]$. Hence the condition (1) is satisfied. Next, for the proof of the condition (2), we define

$$W_{N,\ell}(y) := Q_{N,\ell}(y) Q_{N,\ell}(4y(1-y)) - (Q_{N,\ell}(3/4))^2 \tag{4.11}$$

and verify that for any $\ell = 0, \ldots, N-2$,

$$W_{N,\ell+1}(y) - W_{N,\ell}(y) \leq 0, \quad y \in [3/4, 1]. \tag{4.12}$$

Note here that $Q_{N,0}(y) \equiv 1$, which implies $W_{N,0}(y) \equiv 0$, and that $Q_{N,N-1} = Q_N$ due to the identity $\binom{2N-1}{N-1} = \binom{2N-1}{N}$. Then Proposition 4.4 (2) follows immediately from (4.12). To this end, let us first observe from (4.7) and (4.11) that

$$
\begin{aligned}
&W_{N,\ell+1}(y)-W_{N,\ell}(y)\\
&\quad=Q_{N,\ell+1}(y)Q_{N,\ell+1}(4y(1-y))-Q_{N,\ell}(y)Q_{N,\ell}(4y(1-y))-(Q_{N,\ell+1}(3/4))^2+(Q_{N,\ell}(3/4))^2\\
&\quad=Q_{N,\ell+1}(y)Q_{N,\ell}(4y(1-y))+Q_{N,\ell+1}(y)\nu_{\ell+1}(4y(1-y))^{\ell+1}\left(\frac{\ell+2}{\ell+1}-4y(1-y)\right)\\
&\qquad-Q_{N,\ell}(y)Q_{N,\ell}(4y(1-y))-(Q_{N,\ell+1}(3/4))^2+(Q_{N,\ell}(3/4))^2\\
&\quad=Q_{N,\ell+1}(y)\nu_{\ell+1}(4y(1-y))^{\ell+1}\left(\frac{\ell+2}{\ell+1}-4y(1-y)\right)\\
&\qquad+Q_{N,\ell}(4y(1-y))[Q_{N,\ell+1}(y)-Q_{N,\ell}(y)]-(Q_{N,\ell+1}(3/4))^2+(Q_{N,\ell}(3/4))^2\\
&\quad=Q_{N,\ell+1}(y)\nu_{\ell+1}(4y(1-y))^{\ell+1}\left(\frac{\ell+2}{\ell+1}-4y(1-y)\right)\\
&\qquad+Q_{N,\ell}(4y(1-y))\nu_{\ell+1}y^{\ell+1}\left(\frac{\ell+2}{\ell+1}-y\right)-(Q_{N,\ell+1}(3/4))^2+(Q_{N,\ell}(3/4))^2\\
&\quad=\frac{\ell+2}{\ell+1}\nu_{\ell+1}\left[\Lambda_{\ell+1}(4y(1-y))Q_{N,\ell+1}(y)+\Lambda_{\ell+1}(y)Q_{N,\ell}(4y(1-y))\right]\\
&\qquad-(Q_{N,\ell+1}(3/4))^2+(Q_{N,\ell}(3/4))^2.
\end{aligned}
$$

Since $W_{N,\ell+1}(3/4)-W_{N,\ell}(3/4)=0$, in order to prove the relation (4.12), it suffices to show that $W_{N,\ell+1}(y)-W_{N,\ell}(y)$ decreases monotonically on $[3/4,1]$. Seeing that $Q_{N,\ell+1}(y)\geq Q_{N,\ell}(y)$ for any $y\in[0,1]$, it is equivalent to verify that

$$G(y):=\Lambda_{\ell+1}(4y(1-y))Q_{N,\ell+1}(y)+\Lambda_{\ell+1}(y)Q_{N,\ell}(4y(1-y))$$

decreases monotonically on $[3/4,1]$, i.e., $G'(y)\leq 0$, $y\in[3/4,1]$. Now, we compute G' as follows:

$$
\begin{aligned}
G'(y)=&-(\ell+1)(8y-4)(4y(1-y))^{\ell}[1-4y(1-y)]Q_{N,\ell+1}(y)+\Lambda_{\ell+1}(4y(1-y))Q'_{N,\ell+1}(y)\\
&+(\ell+1)y^{\ell}(1-y)Q_{N,\ell}(4y(1-y))-(8y-4)\Lambda_{\ell+1}(y)Q'_{N,\ell}(4y(1-y)).
\end{aligned}
$$

From (4.7), we find the identity

$$Q'_{N,\ell}(y)=Q'_{N,\ell+1}(y)-(\ell+2)\nu_{\ell+1}y^{\ell}(1-y). \tag{4.13}$$

Also, we see from (4.7) that $Q_{N,\ell+1}(y)=Q_{N,\ell}(y)+\nu_{\ell+1}\frac{\ell+2}{\ell+1}\Lambda_{\ell+1}(y)$. This together with (4.7) and (4.13) implies that

$$
\begin{aligned}
G'(y)=&-(\ell+1)(8y-4)(4y(1-y))^{\ell}[1-4y(1-y)]Q_{N,\ell}(y)+\Lambda_{\ell+1}(4y(1-y))Q'_{N,\ell+1}(y)\\
&+(\ell+1)y^{\ell}(1-y)Q_{N,\ell}(4y(1-y))-(8y-4)\Lambda_{\ell+1}(y)Q'_{N,\ell+1}(4y(1-y)).
\end{aligned}
$$

Using (4.8) and (4.9), a direct calculation shows that

$$G'(y)=\sum_{j=0}^{\ell}\binom{N-1+j}{j}y^j(4y(1-y))^j(f_{j,\ell}(y)+(N+j)g_{j,\ell}(y)), \tag{4.14}$$

where $f_{j,\ell}$ and $g_{j,\ell}$ are defined by

$$\begin{aligned} f_{j,\ell}(y) :=& -(\ell+1)(8y-4)\left(4y(1-y)\right)^{\ell-j}\left(1-4y(1-y)\right)\left(1-\frac{j}{\ell+1}y^{\ell+1-j}\right) \\ &+(\ell+1)y^{\ell-j}(1-y)\left(1-\frac{j}{\ell+1}(4y(1-y))^{\ell+1-j}\right), \\ g_{j,\ell}(y) :=& \Lambda_{\ell+1-j}(4y(1-y))(1-y^{\ell+1-j})-(8y-4)\Lambda_{\ell+1-j}(y)(1-(4y(1-y))^{\ell+1-j}). \end{aligned} \tag{4.15}$$

In order to prove that $G'(y)\le 0$, we show that for $0\le j\le \ell\le N-2$

$$f_{j,\ell}(y)+(N+j)g_{j,\ell}(y)\le 0, \quad \forall y\in[3/4,1]. \tag{4.16}$$

First, to estimate $f_{j,\ell}(y)$, we see that

$$\begin{aligned} f_{j,\ell}(y) =& -(\ell+1)(8y-4)\left(4y(1-y)\right)^{\ell-j}\left(1-4y(1-y)\right)+(\ell+1)y^{\ell-j}\left(1-y\right) \\ &+j\left(\left(4y(1-y)\right)^{\ell-j}y^{\ell-j}\left((8y-4)y\left(1-4y(1-y)\right)-4y(1-y)(1-y)\right)\right). \end{aligned} \tag{4.17}$$

Since $8y-4\ge 2$, $0\le 4y(1-y)\le 3/4$ for $y\in[3/4,1]$, we have

$$(8y-4)y\left(1-4y(1-y)\right)-4y(1-y)(1-y)\ge \frac{3}{16}>0.$$

For $j\le \ell$, it leads to the relation

$$\begin{aligned} f_{j,\ell}(y) \le& -(\ell+1)(8y-4)\left(4y(1-y)\right)^{\ell-j}\left(1-4y(1-y)\right)+(\ell+1)y^{\ell-j}\left(1-y\right) \\ &+(\ell+1)\left(\left(4y(1-y)\right)^{\ell-j}y^{\ell-j}\left((8y-4)y\left(1-4y(1-y)\right)-4y(1-y)(1-y)\right)\right) \\ =& -(\ell+1)(8y-4)\left(4y(1-y)\right)^{\ell-j}\left(1-4y(1-y)\right)\left(1-y^{\ell+1-j}\right) \\ &+(\ell+1)y^{\ell-j}(1-y)\left(1-\left(4y(1-y)\right)^{\ell+1-j}\right). \end{aligned} \tag{4.18}$$

Also, since $0\le 4y(1-y)\le y\le 1$ and $-16y^2+40y-20\ge 0$ for $y\in[3/4,1]$, we obtain

$$\begin{aligned} g_{j,\ell}(y) &\le y^{\ell-j}\left[\Lambda_1(4y(1-y))-(8y-4)\Lambda_1(y)\right]\left(1-\left(4y(1-y)\right)^{\ell+1-j}\right) \\ &= y^{\ell-j+1}\left(1-\left(4y(1-y)\right)^{\ell+1-j}\right)\left(8-12y+\frac{\ell+1}{\ell+2}y\left(-16y^2+40y-20\right)\right) \\ &\le -8y^{\ell-j+1}\left(1-\left(4y(1-y)\right)^{\ell+1-j}\right)(2y-1)(1-y)^2. \end{aligned} \tag{4.19}$$

This implies that $g_{j,\ell}(y)\le 0$. Thus, if $N+j\ge \ell+2+j\ge \ell+2$, $(N+j)g_{j,\ell}(y)\le(\ell+2)g_{j,\ell}(y)$. Putting this and (4.19) into (4.16), we have

$$\begin{aligned} f_{j,\ell}+(N+j)g_{\ell,j}(y) \le& -(\ell+1)(8y-4)\left(4y(1-y)\right)^{\ell-j}\left(1-4y(1-y)\right)(1-y^{\ell+1-j}) \\ &+(\ell+1)y^{\ell-j}\left(1-y\right)\left(1-\left(4y(1-y)\right)^{\ell+1-j}\right) \\ &+(\ell+2)\Lambda_{\ell+1-j}(4y(1-y))(1-y^{\ell+1-j}) \\ &-(\ell+2)(8y-4)\Lambda_{\ell+1-j}(y)(1-(4y(1-y))^{\ell+1-j}) \\ =& \left(4y(1-y)\right)^{\ell-j}\left(1-y^{\ell+1-j}\right)h_1(y)+y^{\ell-j}(1-(4y(1-y))^{\ell+1-j})h_2(y) \end{aligned} \tag{4.20}$$

where h_1 and h_2 are given by

$$h_1(y) := -(\ell+1)(8y-4)\,(1-4y(1-y)) + (\ell+2)\Lambda_1(4y(1-y))$$
$$h_2(y) := (\ell+1)\,(1-y) - (\ell+2)(8y-4)\Lambda_1(y).$$

Since $8y-4 \geq 2$ and $4y(1-y) \leq 1$ on $[3/4,1]$, we have

$$h_1(y) \leq -(\ell+1)2\,(1-4y(1-y)) + (\ell+2)\left(1 - \frac{\ell+1}{\ell+2}\,4y(1-y)\right) \leq 1. \tag{4.21}$$

Moreover, since $(8y-4)y > 1$ on $[3/4,1]$, we have

$$h_2(y) \leq (\ell+1)\,(1-y) - (\ell+2)\left(1 - \frac{\ell+1}{\ell+2}\,y\right) = -1. \tag{4.22}$$

Since $0 \leq 4y(1-y) \leq y \leq 1$ for any $y \in [3/4,1]$, combining (4.21) and (4.22) with (4.20) yields

$$\begin{aligned} f_{j,\ell}(y) + (N+j)g_{\ell,j}(y) &\leq (4y(1-y))^{\ell-j}\left(1-y^{\ell+1-j}\right) - y^{\ell-j}\left(1-(4y(1-y))^{\ell+1-j}\right) \\ &\leq y^{\ell-j}\left(1-y^{\ell+1-j}\right) - y^{\ell-j}\left(1-y^{\ell+1-j}\right) = 0, \end{aligned}$$

which implies $G'(y) \leq 0$, $y \in [3/4,1]$, with $G'(y)$ in (4.14). This completes the proof. □

5 Compactly Supported Biorthogonal Wavelets

5.1 Biorthogonal Wavelet Systems

Let $\phi \in L_2(\mathbb{R})$ be a stable refinable function with the symbol $a(z)$ such that $a(1) = 2$ and $a(-1) = 0$. As usual, the first step for the construction of biorthogonal wavelet systems is to find a refinable function $\tilde{\phi} \in L_2(\mathbb{R})$ such that

$$\langle \phi, \tilde{\phi}(\cdot - \ell)\rangle = \delta_{0,\ell}, \quad \ell \in \mathbb{Z}. \tag{5.1}$$

If $\tilde{\phi}$ is stable and satisfies the condition (5.1), we call $\tilde{\phi}$ the dual refinable function of ϕ (or just dual of ϕ). Let $\tilde{a}(z)$ be the symbol of $\tilde{\phi}$ such that $\tilde{a}(1) = 2$ and $\tilde{a}(-1) = 0$. For convenience, we use the notation

$$m_0(\xi) = a(e^{-i\xi})/2, \quad \tilde{m}_0(\xi) = \tilde{a}(e^{-i\xi})/2.$$

Then, the refinable functions ϕ and $\tilde{\phi}$ are defined respectively by

$$\hat{\phi}(\xi) := \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \hat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi). \tag{5.2}$$

These infinite products in (5.2) converge absolutely and uniformly on compact sets and are the Fourier transforms of compactly supported ϕ and $\tilde{\phi}$ with their support widths given by the filter lengths [5, 8]. A necessary condition for ϕ and $\tilde{\phi}$ to satisfy the duality condition (5.1) is

$$\overline{m_0(\cdot)}\tilde{m}_0(\cdot) + \overline{m_0(\cdot+\pi)}\tilde{m}_0(\cdot+\pi) = 1. \tag{5.3}$$

Given a pair of dual refinable functions ϕ and $\tilde{\phi}$ with their associated filters $m_0(\xi)$ and $\tilde{m}_0(\xi)$, the dual wavelet functions ψ and $\tilde{\psi}$ are defined via the relation

$$\hat{\psi}(\xi) = m_1(\xi/2)\hat{\phi}(\xi/2), \quad \hat{\tilde{\psi}}(\xi) = \tilde{m}_1(\xi/2)\hat{\tilde{\phi}}(\xi/2), \tag{5.4}$$

where

$$m_1(\xi) = e^{-i\xi}\overline{\tilde{m}_0(\xi+\pi)}, \quad \tilde{m}_1(\xi) = e^{-i\xi}\overline{m_0(\xi+\pi)}. \tag{5.5}$$

We usually call $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ a biorthogonal MRA-wavelet system if the following conditions hold: (i) $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ and $\{\tilde{\psi}_{j,k} : j,k \in \mathbb{Z}\}$ are Riesz bases for $L_2(\mathbb{R})$ respectively, and they are biorthogonal in the sense that $\langle \psi_{j,k}, \tilde{\psi}_{\ell,m}\rangle = \delta_{j,\ell}\delta_{k,m}$ with $j,k,\ell,m \in \mathbb{Z}$.; (ii) the condition

$$\langle \phi, \tilde{\psi}(\cdot - \ell)\rangle = \langle \psi, \tilde{\phi}(\cdot - \ell)\rangle = 0 \tag{5.6}$$

is satisfied. In the following theorem, we give a sufficient condition for ψ and $\tilde{\psi}$ to be biorthogonal wavelets associated with ϕ and $\tilde{\phi}$. In fact, this theorem is a slight generalization of Proposition 4.9 in [5] and explained in view of subdivision scheme. Since the proof is almost the same, we abbreviate it here. A reader who is interested in this proof is referred to [14]; this paper is a reduced version of the preprint [14]. For this theorem, it is necessary for a reader to remind that the norm of the iterated scheme S^k, i.e., $\|S^k\|$, is defined in (2.6).

Theorem 5.1 *Let ϕ and $\tilde{\phi}$ be refinable functions whose symbols $a(z)$ and $\tilde{a}(z)$ are respectively of the form*

$$a(z) = \Big(\frac{1+z}{2}\Big)^{\ell} b(z),\ \ \tilde{a}(z) = \Big(\frac{1+z}{2}\Big)^{\tilde{\ell}} \tilde{b}(z) \tag{5.7}$$

for some $\ell, \tilde{\ell} \in \mathbb{N}$, where $b(z)$ and $\tilde{b}(z)$ are Laurent polynomials such that $b(1) = \tilde{b}(1) = 2$. Assume that $\|(\frac{1}{2}S_b)^k\| < 1$ and $\|(\frac{1}{2}S_{\tilde{b}})^{\tilde{k}}\| < 1$ for some $k, \tilde{k} > 0$, where S_b and $S_{\tilde{b}}$ are subdivision schemes associated with $b(z)$ and $\tilde{b}(z)$ respectively. Then, if $\ell + \tilde{\ell} \geq 3$, $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ in (5.2) and (5.4) is a biorthogonal MRA-wavelet system.

5.2 Approximation Order and Vanishing Moment

For $f \in L^2(\mathbb{R})$, we define the biorthogonal projection $P_j f$ of f onto the space V_j by

$$P_j f = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}_{j,k}\rangle \phi_{j,k}. \tag{5.8}$$

For a pair of biorthogonal wavelets ψ and $\tilde{\psi}$, we also define a projection $Q_j f$ of f onto the space $W_j := \overline{\mathrm{span}}\{\psi_{j,k} : k \in \mathbb{Z}\}$ by

$$Q_j f = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}_{j,k}\rangle \psi_{j,k}.$$

By construction, it is obvious that $\|P_j f - f\|_{L_2(\mathbb{R})} = O(2^{-jL})$. In many applications, $P_j f$ is interpreted as an approximation to f at the resolution 2^{-j}, while $Q_j f$ represents the fine detail in f. One advantage of using our biorthogonal wavelet system is as follows. If we use sample values of smooth function as refinable function coefficients at a fine scale, then the resulting biorthogonal projection $P_j f(x)$ approximates the underlying function f with the (optimal) approximation rate $O(2^{-jL})$. More specifically, the value $2^{j/2}\langle f, \tilde{\phi}_{j,k}\rangle$ can be approximated by the function value $f(2^{-j}k)$ with the error bound $O(2^{-jL})$ for $f \in C^L(\mathbb{R})$. The next theorem treats this approximation.

Theorem 5.2 *Let ϕ and $\tilde{\phi}$ be refinable functions obtained from the subdivision scheme S_L with L even and assume that their symbols $a(z)$ and $\tilde{a}(z)$ are respectively of the form in (5.7). Let $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ be a corresponding biorthogonal MRA-wavelet system. Assume that $f \in C^L(\mathbb{R})$. Then, for any fixed $j \in \mathbb{N}$ and $k \in \mathbb{Z}$,*

$$|f(2^{-j}k) - 2^{j/2}\langle f, \tilde{\phi}_{j,k}\rangle| = O(2^{-jL}).$$

Proof. First, by change of variables, it is clear that

$$2^{j/2}\langle f, \tilde{\phi}_{j,k}\rangle = 2^{j/2}\int_{\mathbb{R}} f(t)\tilde{\phi}_{j,k}(t)dt = \int_{\mathbb{R}} f(2^{-j}t)\tilde{\phi}(t-k)dt.$$

Using the identity $\int_{\mathbb{R}} \tilde{\phi}(t-k)dt = 1$ and taking the Taylor polynomial of $f(2^{-j}t)$ of degree $L-1$ at $2^{-j}k$, we get

$$\begin{aligned} f(2^{-j}k) - 2^{j/2}\langle f, \tilde{\phi}_{j,k}\rangle &= \int_{\mathbb{R}} (f(2^{-j}k) - f(2^{-j}t))\tilde{\phi}(t-k)dt \\ &= -\int_{\mathbb{R}} \left(\sum_{n=1}^{L-1} \frac{f^{(n)}(2^{-j}k)}{n!} 2^{-nj}(t-k)^n + R_L f(t)\right)\tilde{\phi}(t-k)dt \end{aligned}$$

where $R_L f$ is the remainder of Taylor expansion

$$R_L f(t) = f^{(L)}(\xi)2^{-jL}(t-k)^L/L!$$

with ξ between $t2^{-j}$ and $k2^{-j}$. By construction $\tilde{\phi}_{j,k}$ reproduces polynomials in $\Pi_{<L}$ (see (5.7)) such that the first integral on the right-hand side of the above equation is identically zero. Thus, it follows that

$$|f(2^{-j}k) - 2^{j/2}\langle f, \tilde{\phi}_{j,k}\rangle| \le c\|f^{(L)}\|_\infty 2^{-jL}\int_{\mathbb{R}} |(t-k)^L\tilde{\phi}(t-k)|dt.$$

Since $\tilde{\phi}$ is compactly supported, $\int_{\mathbb{R}} |(t-k)^L\tilde{\phi}(t-k)|dt < \infty$, which completes the proof. $\square$

Let $\tilde{\psi}_{j,n} := \sum_{k\in\mathbb{Z}} \tilde{d}_{n-2k}\tilde{\phi}_{j+1,k}$. Since $2^{(j+1)/2}\langle f, \tilde{\psi}_{j,k}\rangle = \sum_{k\in\mathbb{Z}} \tilde{d}_{n-2k}\langle f, \phi_{j+1,k}\rangle$, the following corollary holds immediately by Theorem 5.2.

Corollary 5.3 *Let ϕ be the refinable function obtained from the scheme S_L and let $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$ be a corresponding biorthogonal MRA-wavelet system. Assume that $f \in C^L(\mathbb{R})$. Then, for any fixed $j \in \mathbb{N}$ and $k \in \mathbb{Z}$,*

$$|\sum_{k\in\mathbb{Z}} \tilde{d}_{n-2k} f(2^{-j-1}k) - 2^{(j+1)/2}\langle f, \tilde{\psi}_{j,k}\rangle| = O(2^{-(j+1)L}).$$

L	$\pm n$	a_n	$\tilde{a}_n$
	0	$1-2\omega$	$1/2(4\omega-3)/(-1+4\omega)$
2	1	$1/2$	$1/2(-1+5\omega+4\omega^2)/(-1+4\omega)$
	2	ω	$1/4(1+4\omega)/(-1+4\omega)$
	3	0	$-1/2\omega(1+4\omega)/(-1+4\omega)$
	0	$1+6\omega$	$1/64(87+6192\omega+41984\omega^2)/(16\omega+1)/(64\omega+1)$
	1	$9/16$	$-1/16(-9-741\omega+21504\omega^3-9872\omega^2)/(16\omega+1)/(64\omega+1)$
	2	-4ω	$1/256(-63-2992\omega+64512\omega^2)/(16\omega+1)/(64\omega+1)$
4	3	$-1/16$	$1/16(-1-113\omega+33792\omega^3-2128\omega^2)/(16\omega+1)/(64\omega+1)$
	4	ω	$-1/128(9216\omega^2-9-464\omega)/(16\omega+1)/(64\omega+1)$
	5	0	$-1/16\omega(-13+13312\omega^2-528\omega)/(16\omega+1)/(64\omega+1)$
	6	0	$1/256(-80\omega+1024\omega^2-1)/(16\omega+1)/(64\omega+1)$
	7	0	$1/16\omega(-80\omega+1024\omega^2-1)/(16\omega+1)/(64\omega+1)$
	0	$1-20\omega$	$[1/16384(-572427-54787440640\omega^2+1552496459776\omega^3+322878144\omega)$
	1	$150/256$	$3/128(-675-77033920\omega^2+22145925120\omega^4+3424518144\omega^3+406055\omega)$
	2	15ω	$3/32768(66825+2837577728\omega^2+332188876800\omega^3-30911040\omega)$
	3	$-25/256$	$1/256(675+94789248\omega^2-250987151360\omega^4-4279500800\omega^3+452090\omega)$
	4	-6ω	$-3/8192(7425+341966848\omega^2+36909875200\omega^3-3532864\omega)$
6	5	$3/256$	$3/256(-27-7393856\omega^2+53016002560\omega^4+404881408\omega^3+26729\omega)$
	6	ω	$1/65536(41175+2279473152\omega^2+204682035200\omega^3-20958656\omega)$
	7	0	$-3/256\omega(3105+129105920\omega^2-1454144\omega+15435038720\omega^3)$
	8	0	$-3/32768(675+54919168\omega^2+3355443200\omega^3-401600\omega)$
	9	0	$1/256\omega(-690752\omega+5771362304\omega^3+1161+92405760\omega^2)$
	10	0	$9/65536(27+2490368\omega^2-16064\omega+134217728\omega^3)$
	11	0	$-3/256\omega(27+2490368\omega^2-16064\omega+134217728\omega^3)]$
			$/(64\omega-1)/(2097152\omega^2-14336\omega+27)$

Table 2: The dual mask for $L = 2, 4, 6$.

6 Examples

6.1 Dual Refinable Functions

Let $\phi \in L_2(\mathbb{R})$ be a refinable function associated with S_L such that its integer translates are linear independent, that is, ω is chosen to satisfy (3.11). The construction of its dual refinable function $\tilde{\phi}$ usually starts from finding a dual symbol $\tilde{a}(z)$ such that the relation

$$a(z)\overline{\tilde{a}(z)} + a(-z)\overline{\tilde{a}(-z)} = 4, \qquad z = e^{-i\xi}. \tag{6.1}$$

Recall that if $y = \sin^2 \xi/2$,

$$\mathcal{A}(y) := a(e^{i\xi}) = (1-y)^N \left[2 \sum_{n=0}^{N-1} \binom{N-1+n}{n} y^n + \omega 2^{4N} (-1)^N y^N \right].$$

Here, we want to construct the dual symbol $\tilde{a}(z)$ which has the factor $(1-y)^N$. Thus, letting

$$P_1(y) = (1-y)^N \mathcal{A}(y)/2,$$

the problem to find the dual $\tilde{a}(z)$ in (6.1) is equivalent to constructing $P_2(y)$ which solves the Bezout problem

$$P_1(y)P_2(y) + P_1(1-y)P_2(1-y) = 1 \tag{6.2}$$

where the degree of $P_2(y)$ is $3N-1$; see ([7]) for the details of the Bezout problem. Since $\phi(\cdot - k)$, $k \in \mathbb{Z}$, are linear independent, it is immediate from Theorem 3.7 that there is no common zero of $P_1(y)$ and $P_2(y)$, which guarantees the existence of $P_2(y)$. Then we obtain the dual symbol $\tilde{a}(z)$ such that

$$\tilde{a}(e^{i\xi}) = 2(1-y)^N P_2(y), \quad y = \sin^2 \xi/2.$$

For $L = 2,\ 4$ and 6, the specific forms of the dual mask of $\{\tilde{a}_n : n \in \mathbb{Z}\}$ are given in Table 2. Some examples of biorthogonal wavelet systems $L = 4$ are computed with respect to several different choices of ω. In this case, the refinable functions reproduce cubic polynomials and the wavelet functions have the vanishing moment of order 4. Eventually, it becomes the Coifman biorthogonal wavelet of order 4. Figure 1 indicates that the dual functions ϕ, $\tilde{\phi}$ and their associated wavelets ψ, $\tilde{\psi}$ for $\omega = 0.025, 0, -0.005, -0.0203$. In particular, if $\omega = 0$, it becomes the minimal length Coiffman biorthogonal wavelet system.

6.2 Dual Functions with Less Dissimilar Lengths

We are concerned with the biorthogonal wavelet systems of less dissimilar filter lengths. Let $L = 2N$. Involving the equation of Bezout problem in (6.2), let the zeros of $P_2(y)$ are $\lambda_m, \bar{\lambda}_m \in \mathbb{C}$ with $m = 1, \dots, K$ and $y_n \in \mathbb{R}$ with $n = 1, \dots, 3N - 2K - 1$. Then $P_2(y)$ is factored into the form

$$P_2(y) = C \prod_{n=1}^{3N-2K-1} (y - y_n) \prod_{m=1}^{K} (y^2 - 2y\mathrm{Re}\lambda_m + |\lambda_m|^2) \tag{6.3}$$

for some constant C. On the purpose of constructing new refinable dual functions ϕ and $\tilde{\phi}$, we regroup the factors of $P_2(y)$ in (6.3). For this, we set

$$\begin{aligned} h(e^{i\xi}) &:= a(e^{i\xi}) C_1 \prod_{\ell \in I_1} (y - y_\ell) \prod_{m \in J_1} (y^2 - 2y\mathrm{Re}\lambda_m + |\lambda_m|^2) \\ g(e^{i\xi}) &:= (1-y)^N C_2 \prod_{\ell \in I_2} (y - y_\ell) \prod_{m \in J_2} (y^2 - 2y\mathrm{Re}\lambda_m + |\lambda_m|^2), \end{aligned}$$

where $y = \sin^2 \xi/2$ and

$$C = C_1 C_2, \quad I_1 \cup I_2 = \{1, \dots, 3N - 1 - 2K\}, \quad J_1 \cup J_2 = \{1, \dots, K\}.$$

Then, we can derive new refinable functions ϕ and $\tilde{\phi}$ with the new symbols h and g.

Example 6.1 (13-11 Tab biorthogonal wavelet system based on cubic) *Let $L = 4$. Then, by using symbolic computation with MAPLE 8, we obtain*

$$\begin{aligned} P_2(y) =& 2 + 4y + (12 - 256\omega)y^2 + 16y^3 + \frac{16(98304\omega^3 - 6656\omega^2 - 176\omega - 1)}{1 + 80\omega + 1024\omega^2} y^4 \\ &- \frac{1024\omega(-1 - 80\omega + 1024\omega^2)}{1 + 80\omega + 1024\omega^2} y^5. \end{aligned}$$

Let $y_1(\neq 0)$ be a real root of $P_2(y)$. Here $y_1 \in \mathbb{R}$, and t_1, t_2, t_3, t_4 can be real or complex numbers depending on the value of ω. Then, for $y = \sin^2 \xi/2$, we have

$$h(e^{i\xi}) = 2(1-y)^2(1+2y+256\omega y^2)(y-y_1)/y_1$$
$$g(e^{i\xi}) = 2(1-y)^2 P_2(y) y_1/(y-y_1).$$

Figure 7 indicates the dual functions ϕ, $\tilde{\phi}$ and their associated wavelets ψ, $\tilde{\psi}$ with the values $\omega = 0.02, 0, -0.005, -0.015$.

7 Concluding Remarks

In this study, we introduced a large class of biorthogonal wavelets. Each refinement mask has a tension parameter ω. Our initial numerical test observed that by choosing ω in some suitable range, the new biorthogonal wavelets systems have competitive approximation and compression ability (especially for smooth signals) to the Coiffman biorthogonal wavelet system as well as the 9-7 tab biorthogonal wavelet system, which is the most widely used one in the field of wavelet transform coding. Thus, our next project will focus on implementing suitable algorithms for the applications of biorthogonal wavelet systems that balance and meet various demands, such as regularity of wavelets, shapes of refinable functions and approximation power, in time-frequency analysis. On the other hand, one may interested in continuing this study for the case L is odd, i.e., $L = 2N+1$. The masks of S_L are obtained by solving the linear system

$$p_\ell(1/4+\tau/2) = \sum_{n=-N}^{N+\tau} a_{\tau-2n} p_\ell(n), \quad \ell = 1, \ldots, L,$$

where p_ℓ, $\ell = 1, \ldots, L$, is a basis of $\Pi_{<L}$. For the even mask $\{a_{2n} : n \in \mathbb{Z}\}$, setting $a_{-2N-2} = (-1)^N \omega$, their explicit forms are

$$a_{-2n} \;=\; \binom{4N}{2N}\binom{2N}{N+n}\frac{(-1)^n(4N+1)}{(1-4n)4^{3N}} - \omega(-1)^{2N-n}\binom{2N+1}{N+n}$$

for $n = -N, \ldots, N$. For the odd mask $\{a_{1+2n} : n \in \mathbb{Z}\}$, setting $a_{1+2N} = (-1)^N \omega$, their explicit forms are

$$a_{1-2n} \;=\; \binom{4N}{2N}\binom{2N}{N-n+1}\frac{(-1)^{1-n}(4N+1)}{(4n-3)4^{3N}} - \omega(-1)^{2N+n-1}\binom{2N+1}{N-n+1}$$

for $n = -N+1, \ldots, N+1$. However, we leave the corresponding analysis as a next study.

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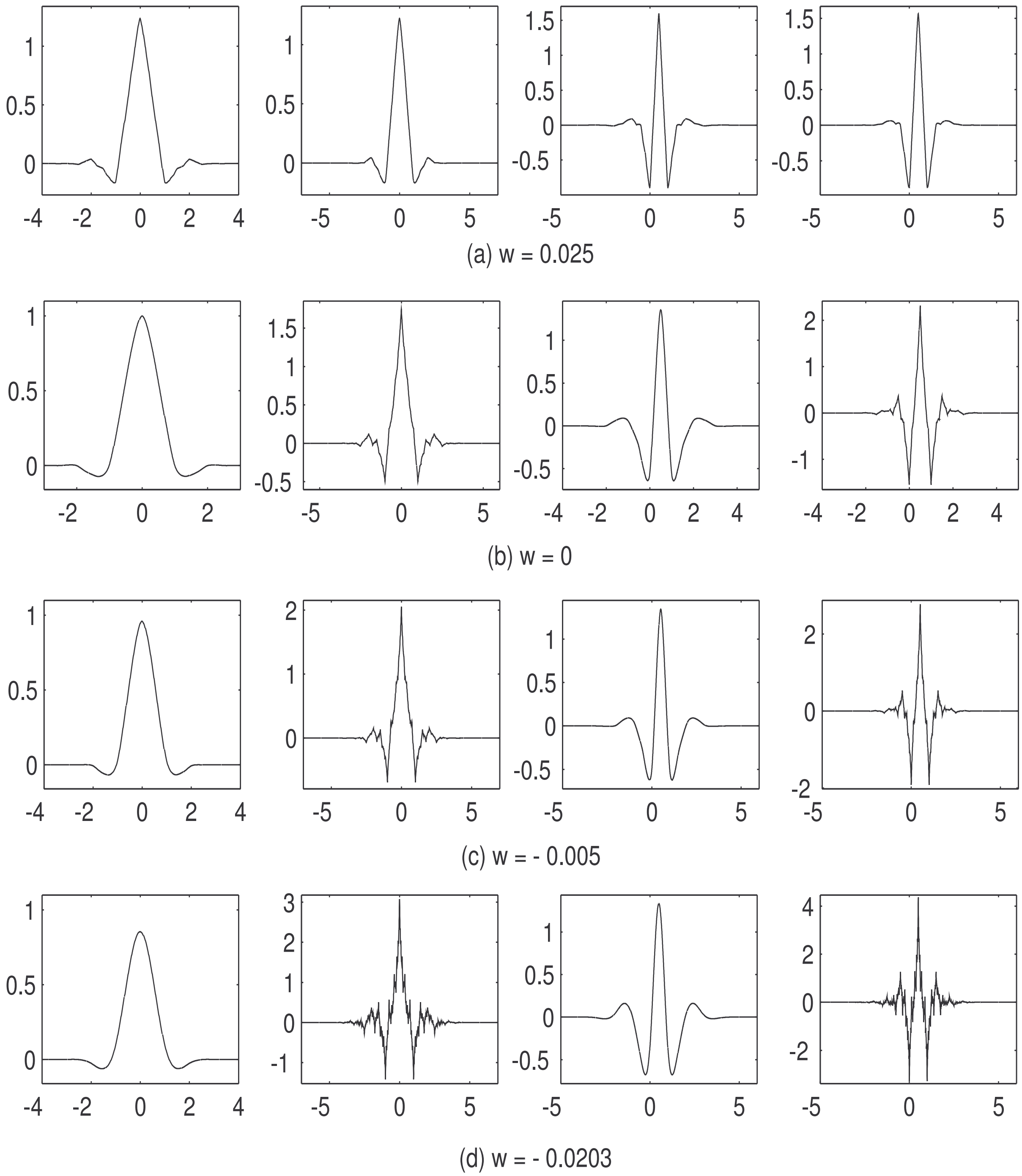

Figure 1: The biorthogonal wavelet systems based on cubic polynomial. From the left, the pictures in each column indicate the refinable functions ϕ, $\tilde{\phi}$ and wavelets ψ, $\tilde{\psi}$. From the top, ω is chosen to be $0.025,\ 0,\ -0.005,\ -0.0203$. If $\omega = 0$, it becomes the minimal length Coiffman biorthogonal wavelet system.

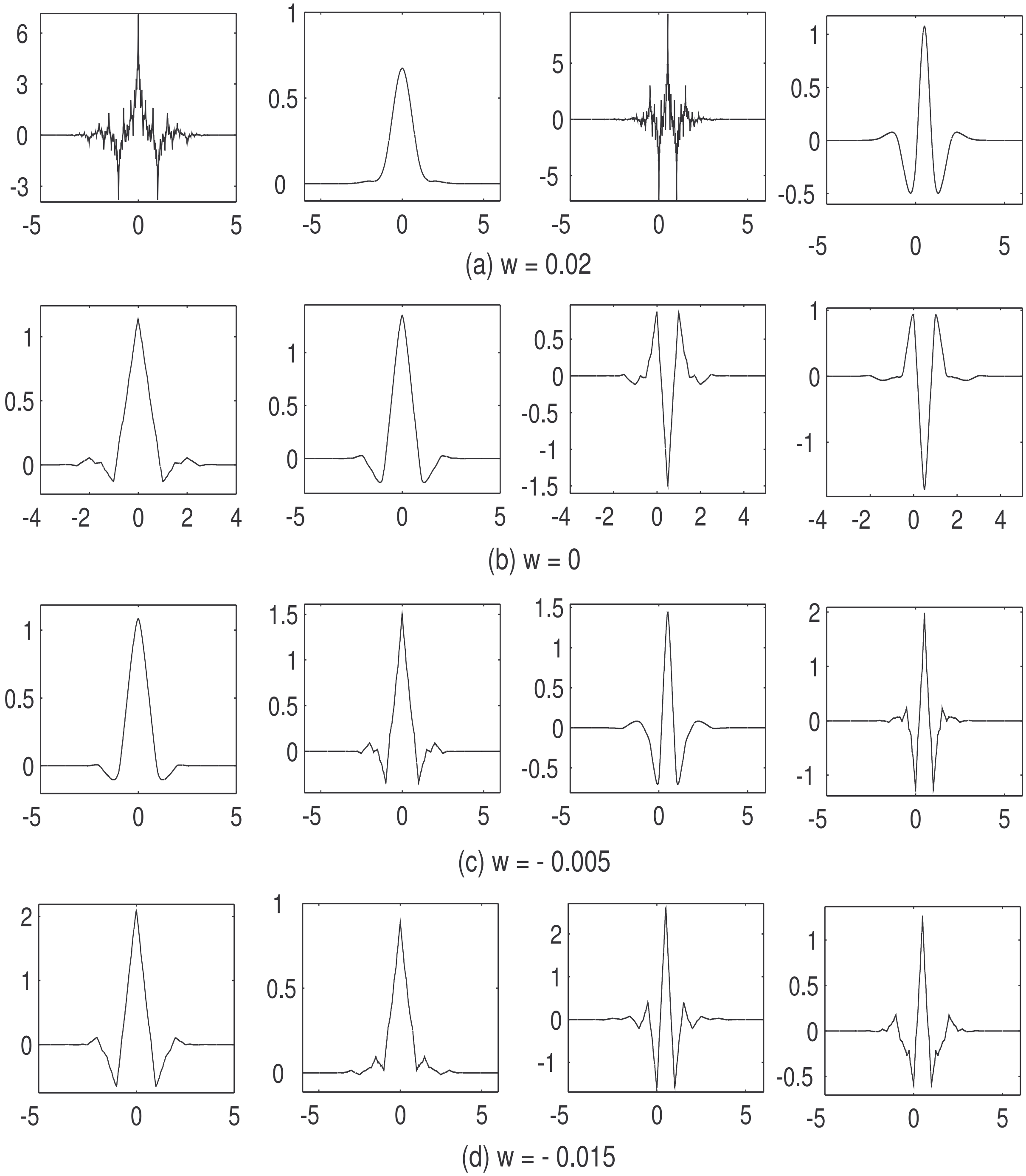

Figure 2: The 9-7 tab biorthogonal wavelet system based on cubic polynomial. From the left, the pictures in each column indicate the refinable functions ϕ, $\tilde{\phi}$ and wavelets ψ, $\tilde{\psi}$. From the top, ω is chosen to be $0.02,\ 0,\ -0.005,\ -0.015$.

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