Tight matrix-generated Gabor frames in $L^2(\mathbb{R}^d)$ with desired time-frequency localization*

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February 12, 2008

Abstract

Based on two real and invertible $d \times d$ matrices B and C such that the norm $||C^T B||$ is sufficiently small, we provide a construction of tight Gabor frames $\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ with explicitly given and compactly supported generators. The generators can be chosen with arbitrary polynomial decay in the frequency domain.

2000 Mathematics Subject Classification: 42C15, 42C40.
Key words and phrases: Gabor frames, Tight frames, Parseval frames.

*This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2006-331-C00014).

1 Introduction

The purpose of this paper is to present a construction of a class of tight matrix-generated Gabor frames in $L^2(\mathbb{R}^d)$. In particular, we focus on construction of frames with explicitly given generators and good time-frequency localization.

The question of construction of tight Gabor frames was first treated in the seminal paper [4] by Daubechies, Grossmann and Meyer, which was dealing with the one-dimensional case. Theoretical results in higher dimensions (i.e., characterization of tight Gabor frames) were obtained in [10] and [6]. Note that non-tight Gabor frames with explicitly given dual generators were constructed in [2] and [3]; the constructions in [3] work in any dimensions, but the expression for the dual generator involves some book-keeping in high dimensions.

In the rest of the introduction, we collect some basic definitions and conventions.

For $y \in \mathbb{R}^d$, the translation operator T_y acting on $f \in L^2(\mathbb{R}^d)$ is defined by

$$(T_y f)(x) = f(x-y), \quad x \in \mathbb{R}^d.$$

For $y \in \mathbb{R}^d$, the modulation operator E_y is

$$(E_y f)(x) = e^{2\pi i y\cdot x} f(x), \quad x \in \mathbb{R}^d,$$

where $y \cdot x$ denotes the inner product between y and x in $\mathbb{R}^d$. Given two real and invertible $d \times d$ matrices B and C and a function $g \in L^2(\mathbb{R}^d)$ we consider Gabor systems of the form

$$\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d} = \{e^{2\pi i Bm\cdot x} g(x-Cn)\}_{m,n\in\mathbb{Z}^d}.$$

The dilation operator associated with a matrix C is

$$(D_C f)(x) = |\det C|^{1/2} f(Cx), \quad x \in \mathbb{R}^d.$$

Let C^T denote the transpose of a matrix C; then

$$D_C E_y = E_{C^T y} D_C, \quad D_C T_y = T_{C^{-1}y} D_C.$$

If C is invertible, we use the notation

$$C^\sharp = (C^T)^{-1}.$$

Furthermore, the norm of a matrix C is defined by

$$||C|| = \sup_{||x||=1} ||Cx||.$$

For $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we denote the Fourier transform by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \gamma} dx.$$

As usual, the Fourier transform is extended to a unitary operator on $L^2(\mathbb{R}^d)$. The reader can check that

$$\mathcal{F} T_{Ck} = E_{-Ck} \mathcal{F}.$$

Recall that a countable family of vectors $\{f_k\}_{k \in I}$ belonging to a separable Hilbert space $\mathcal{H}$ is a *Parseval frame* if

$$\sum_{k \in I} |\langle f, f_k \rangle|^2 = ||f||^2, \ \forall f \in \mathcal{H}.$$

Parseval frames are also known as tight frames with frame bound equal to one. Like orthonormal bases, a Parseval frame provides us with an expansion of the elements in $\mathcal{H}$: in fact, if $\{f_k\}_{k \in I}$ is a Parseval frame, then

$$f = \sum_{k \in I} \langle f, f_k \rangle f_k, \ \forall f \in \mathcal{H}.$$

On the other hand, the conditions for being a Parseval frame is considerably weaker than the condition for being an orthonormal basis; thus, Parseval frames yield more flexible constructions.

Our starting point is a characterization of Parseval frames with Gabor structure; several versions of this result exist in the literature, see [10], [7], [6], [3].

Lemma 1.1 *A family $\{E_{Bm} T_{Cn} g\}_{m,n \in \mathbb{Z}^d}$ forms a Parseval frame for $L^2(\mathbb{R}^d)$ if and only if*

$$\sum_{k \in \mathbb{Z}^d} \overline{g(x - B^\sharp n - Ck)} g(x - Ck) = |\det B| \delta_{n,0}, \ \ a.e. x \in \mathbb{R}^d. \tag{1}$$

2 The results

We now present the first version of our results. We are mainly interested in generators g, whose $\mathbb{Z}^d$–translates form a partition of unity, but we state the result under a weaker assumption. For simplicity we first consider the case $C = I$.

Theorem 2.1 *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R}^d)$ be a non-negative function with supp $g \subseteq [0, N]^d$, for which*

$$\sum_{n \in \mathbb{Z}^d} g(x - n) > 0, \ \ a.e.\ x \in \mathbb{R}^d.$$

Assume that the $d \times d$ matrix B is invertible and $||B|| \leq \frac{1}{\sqrt{d}\ N}$. Define $h \in L^2(\mathbb{R}^d)$ by

$$h(x) := \sqrt{|\det B| \frac{g(x)}{\sum_{n \in \mathbb{Z}^d} g(x-n)}}. \tag{2}$$

Then the function h generates a Parseval frame $\{E_{Bm}T_n h\}_{m,n \in \mathbb{Z}^d}$ for $L^2(\mathbb{R}^d)$.

Proof. Note that

$$0 \leq h \leq \sqrt{|\det B|}\chi_{[0,N]^d};$$

this implies that $h \in L^2(\mathbb{R}^d)$.

We now apply Lemma 1.1. Since B is invertible, for any $n \in \mathbb{Z}^d$ we have

$$|n| = ||B^T B^\sharp n|| \leq ||B||\ ||B^\sharp n||;$$

thus, for $n \neq 0$, $||B^\sharp n|| \geq 1/||B||$. Hence (1) is satisfied for $n \neq 0$ if $1/||B|| \geq \sqrt{d}\ N$, i.e., if

$$||B|| \leq \frac{1}{\sqrt{d}\ N}.$$

For $n = 0$, (1) follows from the the definition (2). □

The construction in Theorem 2.1 has several attractive features: it is given explicitly, and it has compact support. Furthermore, polynomial decay of the generator g of any given order in the frequency domain can be achieved by requiring g to be sufficiently smooth:

Lemma 2.2 *Let $k \in \mathbb{N}$ and let $f \in C^{dk}(\mathbb{R}^d)$ be compactly supported. Then*

$$|\hat{f}(\gamma)| \leq A(1+|\gamma|^2)^{-k/2}.$$

Proof. Note that f is in $L^2(\mathbb{R}^d)$. Integration by parts for a variable x_j implies

$$\begin{aligned}\hat{f}(\gamma) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\cdot\gamma}dx \\ &= \frac{1}{2\pi i\gamma_j}\int_{-\infty}^{\infty}\frac{\partial f}{\partial x_j}\ e^{-2\pi i x\cdot\gamma}dx\end{aligned}$$

Inductively, since f has partial derivative of order kd, we have

$$\begin{aligned}|\hat{f}(\gamma)| &= \left|\frac{1}{\prod_{j=1}^d (2\pi i\gamma_j)^k}\int_{-\infty}^{\infty}\frac{\partial^{kd} f}{\partial x_1^k\cdots\partial x_d^k}\ e^{-2\pi i x\cdot\gamma}dx\right| \\ &\leq \frac{A}{\prod_{j=1}^d (1+|\gamma_j|)^k} \\ &= \frac{A}{\left(\prod_{j=1}^d (1+|\gamma_j|)^2\right)^{k/2}}.\end{aligned}$$

A direct calculation shows that

$$\begin{aligned}\prod_{j=1}^d (1+|\gamma_j|)^2 &\geq (1+|\gamma_1|^2)(1+|\gamma_2|^2)\cdots(1+|\gamma_d|^2) \\ &\geq (1+|\gamma_1|^2+|\gamma_2|^2)(1+|\gamma_3|)^2\cdots(1+|\gamma_d|)^2 \\ &\geq \cdots \\ &\geq 1+|\gamma|^2.\end{aligned}$$

This implies that

$$|\hat{f}(\gamma)| \leq A(1+|\gamma|^2)^{-k/2}.$$

□

Via a change of variable Theorem 2.1 leads to a construction of frames of the type $\{E_{Bm}T_{Cn}h\}_{m,n\in\mathbb{Z}^d}$:

Theorem 2.3 *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R}^d)$ be a non-nenegative function with supp $g \subseteq [0, N]^d$, for which*

$$\sum_{n \in \mathbb{Z}^d} g(x-n) > 0 \quad \textit{for a.e. } x \in \mathbb{R}^d.$$

Let B and C be invertible $d \times d$ matrices such that $||C^T B|| \leq \frac{1}{\sqrt{d}\ N}$, and let

$$h(x) := \sqrt{|\det(CB)| \frac{g(x)}{\sum_{n \in \mathbb{Z}^d} g(x-n)}}. \tag{3}$$

Then the function $D_{C^{-1}}h$ generates a Parseval frame $\{E_{Bm}T_{Cn}D_{C^{-1}}h\}_{m,n \in \mathbb{Z}^d}$ for $L^2(\mathbb{R}^d)$.

Proof. By assumptions and Theorem 2.1, the Gabor system $\{E_{C^T Bm}T_n h\}_{m,n \in \mathbb{Z}^d}$ forms a tight frame; since

$$D_{C^{-1}}E_{C^T Bm}T_n = E_{Bm}T_{Cn}D_{C^{-1}},$$

the result follows from $D_{C^{-1}}$ being unitary. $\square$

We are particulary interested in the case where the integer-translates of the function g generates a partition of unity, i.e.,

$$\sum_{n \in \mathbb{Z}^d} g(x-n) = 1 \quad \text{for } a.e. \ x \in \mathbb{R}^d.$$

In that case, the generator in Theorem 2.3 takes the form

$$D_{C^{-1}}h(x) = \sqrt{|\det(B)|\, g(C^{-1}x)}.$$

Let B_N denote the Nth cardinal B-spline on $\mathbb{R}$, and define the box-spline

$$\mathcal{B}_N(x) = \prod_{i=1}^{d} B_N(x_i), \ \ x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Then

$$\sum_{n \in \mathbb{Z}^d} \mathcal{B}_N(x-n) = 1.$$

Thus, we obtain the following consequence of Theorem 2.3:

Corollary 2.4 *Let $N \in \mathbb{N}$, and let B and C be invertible $d \times d$ matrices such that $||C^T B|| \leq \frac{1}{\sqrt{d}\ N}$. Let*

$$\varphi(x) = \sqrt{|\det(B)|\, B_N(C^{-1}x)}.$$

Then $\{E_{Bm}T_{Cn}\varphi\}_{m,n\in\mathbb{Z}^d}$ is a Parseval frame for $L^2(\mathbb{R}^d)$.

Example 2.5 The one-dimensional B-spline of order 4 is given by

$$B_4(x) = \begin{cases} \frac{x^3}{6}, & x \in [0,1[; \\ \frac{2}{3} - 2x + 2x^2 - \frac{x^3}{2}, & x \in [1,2[; \\ -\frac{22}{3} + 10x - 4x^2 + \frac{x^3}{2}, & x \in [2,3[; \\ \frac{32}{3} - 8x + 2x^2 - \frac{x^3}{6}, & x \in [3,4[; \\ 0, & x \notin [0,4[. \end{cases}$$

Define the box-spline

$$B_4(x) := B_4(x_1)B_4(x_2),\ x = (x_1, x_2) \in \mathbb{R}^2.$$

Let 2×2 matrices B and C be defined by

$$B = \frac{1}{80}\begin{pmatrix} 1 & 6 \\ -2 & 4 \end{pmatrix},\ C = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}.$$

A direct calculation shows that

$$\begin{aligned} ||C^T B|| &= \left|\left|\frac{1}{20}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\right|\right| = \sup_\theta \left|\left|\frac{1}{20}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right|\right| \\ &= \left(\frac{\sqrt{2}}{10}\right). \end{aligned}$$

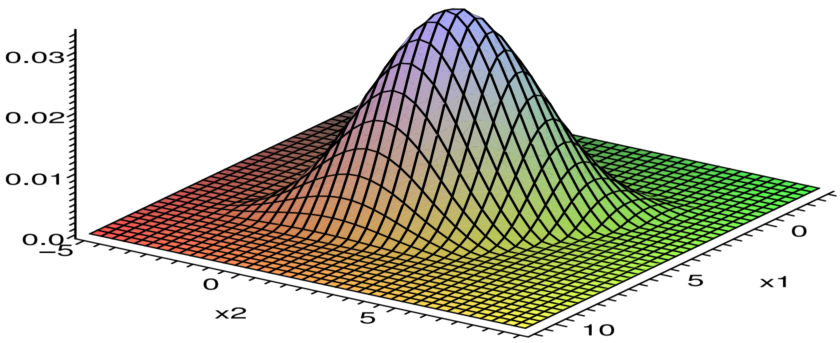
Thus

$$||C^T B|| N\sqrt{d} = \frac{\sqrt{2}}{10}\ 4\sqrt{2} = 0.8 \leq 1.$$

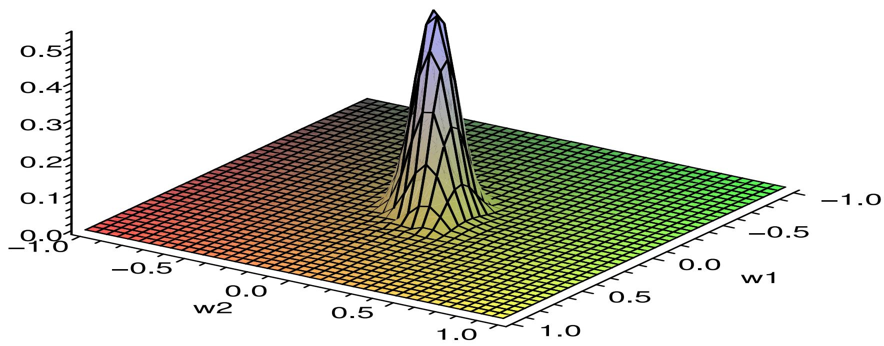
Let

$$\varphi(x) = \sqrt{|\det(B)|\, B_4(C^{-1}x)}.$$

By Corollary 2.4, $\{E_{Bm}T_{Cn}\varphi\}_{m,n\in\mathbb{Z}^2}$ is a Parseval frame for $L^2(\mathbb{R}^2)$. On Figure 1, we plot the functions φ and $|\hat{\varphi}|$.



(a)



(b)

Figure 1: The functions φ (Figure (a)) and $|\hat{\varphi}|$ (Figure (b)) in Example 2.5.

For functions g of the type considered in Theorem 2.3 and arbitrary real invertible $d \times d$ matrices B and C, Theorem 2.3 leads to a construction of a (finitely generated) tight multi–Gabor frame $\{E_{Bm}T_{Cn}h_k\}_{m,n\in\mathbb{Z}^d,k\in\mathcal{F}}$, where all the generators h_k are dilated and translated versions of h:

Theorem 2.6 *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R}^d)$ be a non-negative function with supp $g \subseteq [0, N]^d$, for which*

$$\sum_{n\in\mathbb{Z}^d} g(x-n) = 1.$$

Let B and C be invertible $d \times d$ matrices and choose $J \in \mathbb{N}$ such that $J \geq ||C^T B|| \sqrt{d}\, N$. Define the function h by (3). Then the functions

$$h_k = T_{\frac{1}{J}Ck} D_{JC^{-1}} h, \quad k \in \mathbb{Z}^d \cap [0, J-1]^d$$

generate a multi-Gabor Parseval frame $\{E_{Bm}T_{Cn}h_k\}_{m,n\in\mathbb{Z}^d,k\in\mathbb{Z}^d\cap[0,J-1]^d}$ for $L^2(\mathbb{R}^d)$.

Proof. The choice of J implies that the matrices B and $\frac{1}{J}C$ satisfy the conditions in Theorem 2.3; thus

$$\{e^{2\pi i Bm\cdot x}(D_{JC^{-1}}h)(x - \frac{1}{J}Cn)\}_{m,n\in\mathbb{Z}^d}$$

forms a tight Gabor frame for $L^2(\mathbb{R}^d)$. Now,

$$\left\{\frac{1}{J}Cn\right\}_{n\in\mathbb{Z}^d} = \bigcup_{k\in\mathbb{Z}^d\cap[0,J-1]^d} \left\{\frac{1}{J}Ck + Cn\right\}_{n\in\mathbb{Z}^d}.$$

Thus

$$\begin{aligned}\left\{(D_{JC^{-1}}h)(\cdot - \frac{1}{J}Cn)\right\}_{n\in\mathbb{Z}^d} &= \bigcup_{k\in\mathbb{Z}^d\cap[0,J-1]^d} \left\{(D_{JC^{-1}}h)(\cdot - \frac{1}{J}Ck - Cn)\right\}_{n\in\mathbb{Z}^d} \\ &= \bigcup_{k\in\mathbb{Z}^d\cap[0,J-1]^d} \left\{T_{Cn}T_{\frac{1}{J}Ck}D_{JC^{-1}}h(\cdot)\right\}_{n\in\mathbb{Z}^d}.\end{aligned}$$

Inserting this into the expression for the tight frame leads to the result. $\square$

Example 2.7 Let B_4 be the 4th box-spline in $\mathbb{R}^2$ as in Example 2.5 and let 2×2 matrices B and C be defined by

$$B = \frac{1}{40}\begin{pmatrix} 1 & 6 \\ -2 & 4 \end{pmatrix}, \; C = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}.$$

Then

$$||C^T B|| N\sqrt{d} = \frac{\sqrt{2}}{5}\, 4\sqrt{2} = 1.6 \leq 2.$$

Thus we can apply Theorem 2.6 with $J = 2$. Define

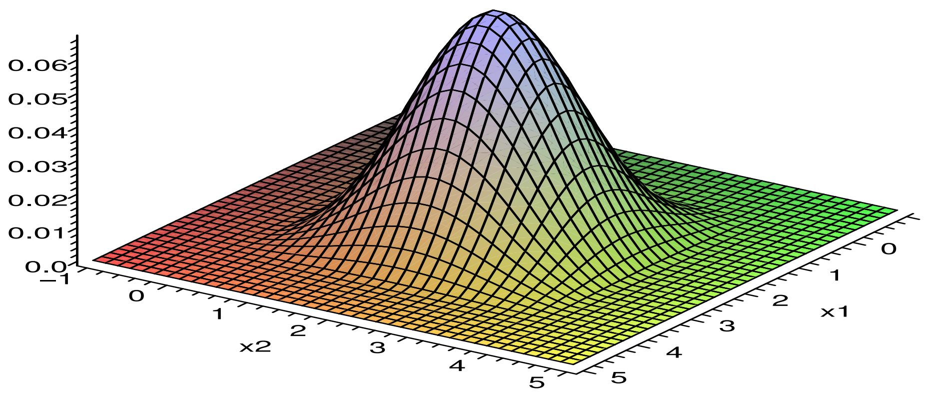
$$h(x) := \sqrt{|\det(CB)| B_4(x)}.$$

By Theorem 2.6, the four functions

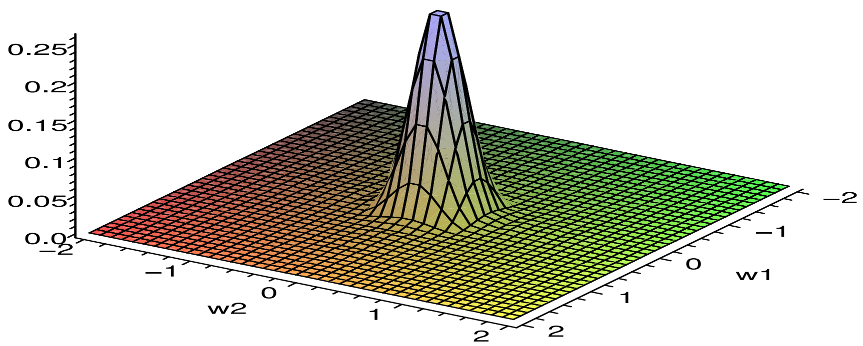
$$h_k = T_{\frac{1}{2}Ck} D_{2C^{-1}} h, \quad k \in \mathbb{Z}^2 \cap [0,1]^2$$

generate a multi-Gabor Parseval frame $\{E_{Bm} T_{Cn} h_k\}_{m,n \in \mathbb{Z}^2, k \in \mathbb{Z}^2 \cap [0,1]^2}$ for $L^2(\mathbb{R}^2)$. On Figure 2, we plot the functions h and $|\hat{h}|$.

Acknowledgment: The authors thank Professor Hong Oh Kim for organizing the conference *International Workshop on Wavelet Frames* at KAIST, South Korea, November 24–28, 2005, and hereby making the work on the present paper possible.



(a)



(b)

Figure 2: The functions h (Figure (a)) and $|\hat{h}|$ (Figure (b)) in Example 2.7.

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