Sobolev exponents of Butterworth refinable functions *

Hong Oh Kim†, Rae Young Kim‡

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Abstract

The precise Sobolev exponent $s_\infty(\varphi_n)$ of the Butterworth refinable function φ_n associated with the Butterworth filter of order n, $b_n(\xi) := \frac{\cos^{2n}(\xi/2)}{\cos^{2n}(\xi/2)+\sin^{2n}(\xi/2)}$, is shown to be $s_\infty(\varphi_n) = n\log_2 3 + \log_2(1+3^{-n})$. This recovers the previously given asymptotic estimate of $s_\infty(\varphi_n)$ of Fan and Sun [1], and gives more accurate regularity of Butterworth refinable function φ_n.

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The Sobolev exponent $s_\infty(f)$ of a function f is defined in terms of its Fourier transform

$$s_\infty(f) = \sup\{s \mid \sup_\xi |\hat{f}(\xi)|(1+|\xi|)^s < \infty\}.$$

This gives the regularity of f as $f \in C^s$ for any $s < s_\infty(f) - 1$.

The Butterworth filter of order n is defined by

$$b_n(\xi) := \cos^{2n}(\xi/2)\mathcal{L}_n(\xi),$$

where

$$\mathcal{L}_n(\xi) := \frac{1}{\cos^{2n}(\xi/2) + \sin^{2n}(\xi/2)}.$$

Then the corresponding refinable function φ_n, called Butterworth refinable function, is given by

$$\begin{aligned}\hat{\varphi}_n(\xi) &:= \prod_{j=1}^\infty b_n(2^{-j}\xi) = \prod_{j=1}^\infty \cos^{2n}(2^{-j-1}\xi) \prod_{j=1}^\infty \mathcal{L}_n(2^{-j}\xi) \\ &= \left(\frac{\sin(\xi/2)}{\xi/2}\right)^{2n} \prod_{j=1}^\infty \mathcal{L}_n(2^{-j}\xi).\end{aligned}$$

Fan and Sun [1] obtained the estimate

$$n\log_2 3 \le s_\infty(\varphi_n) \le n\log_2 3 + \log_2(1+3^{-n}).$$

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†Department of Mathematical Sciences, KAIST, 373-1, Guseong-dong, Yuseong-gu, Daejeon, 305-701, Republic of Korea (hkim@amath.kaist.ac.kr)

‡Department of Mathematics, Yeungnam University, 214-1, Dae-dong, Gyeongsan-si, Gyeongsangbuk-do, 712-749, Republic of Korea (rykim@ynu.ac.kr)

We prove here that the precise Sobolev exponent is their upper bound of $s_\infty(\varphi_n)$:

$$s_\infty(\varphi_n) = n \log_2 3 + \log_2(1 + 3^{-n}).$$

As an application, we also give the precise Sobolev exponents of the special class of refinable orthonormal cardinal functions from Blaschke products in [2].

We recall a method to estimate the decay of $\hat{\varphi}$ of a refinable function φ adapted for our particular purpose in the following proposition. See [3, Lemma 7.1.5, Lemma 7.1.6].

Proposition 1 *For $\mathcal{L} \in C^1(\mathbb{T})$, let b be the refinable filter of the refinable function φ of the form*

$$|b(\xi)| = \cos^{2n}(\xi/2)|\mathcal{L}(\xi)|, \ \xi \in [-\pi, \pi].$$

Suppose that $[-\pi, \pi] = D_1 \cup D_2 \cup D_3$ and that

$$\begin{aligned}
&|\mathcal{L}(\xi)| \le |\mathcal{L}(\frac{2\pi}{3})|, \ \xi \in D_1; \\
&|\mathcal{L}(\xi)\mathcal{L}(2\xi)| \le |\mathcal{L}(\frac{2\pi}{3})|^2, \ \xi \in D_2; \\
&|\mathcal{L}(\xi)\mathcal{L}(2\xi)\mathcal{L}(4\xi)| \le |\mathcal{L}(\frac{2\pi}{3})|^3, \ \xi \in D_3.
\end{aligned}$$

Then

$$|\hat{\varphi}(\xi)| \le C(1 + |\xi|)^{-2n+\kappa},$$

where $\kappa = \log_2(|\mathcal{L}(2\pi/3)|)$, and this decay is optimal; i.e., $s_\infty(\varphi) = 2n - \kappa$. Consequently, $\varphi \in C^s$ for any $s < 2n - \kappa - 1$.

The idea is to divide the interval $[-1/2, 1/2]$ into union of three sets to have the relevant estimates on each set as in the following lemma.

Lemma 2 *Let $Q_n(x) := (1/2 - x)^n + (1/2 + x)^n$. Then*

(a) $Q_n(x) \ge Q_n(\frac{1}{4}), \ x \in [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$;

(b) $Q_n(x)Q_n(1/2 - 4x^2) \ge (Q_n(\frac{1}{4}))^2, \ x \in [-\frac{1}{4}, -\frac{1}{10}] \cup [\frac{1}{10}, \frac{1}{4}]$;

(c) $Q_n(x)Q_n(1/2 - 4x^2)Q_n(-64x^4 + 16x^2 - 1/2) \ge (Q_n(\frac{1}{4}))^3, \ x \in [-\frac{1}{10}, \frac{1}{10}]$.

Proof. Note that $Q_n(x), Q_n(1/2 - 4x^2)$ and $Q_n(-64x^4 + 16x^2 - 1/2)$ are symmetric about the origin. Thus we only assume that $x \in [0, \frac{1}{2}]$.

The condition (a) follows from the fact that Q_n is increasing on $[0, \frac{1}{2}]$, since

$$Q_n'(x) = n(-(1/2 - x)^{n-1} + (1/2 + x)^{n-1}) \ge 0, \ x \in [0, \frac{1}{2}].$$

We now prove the condition (b). For $n = 1$, we have

$$Q_1(x)Q_1(\frac{1}{2} - 4x^2) = 1 = (Q_1(\frac{1}{4}))^2.$$

For $n = 2$, let $f(x) := Q_2(x)Q_2(\frac{1}{2} - 4x^2)$. Then a direct calculation shows that for $0 < x < \frac{1}{4}$,

$$f'(x) = 4x(-1 + 96x^4) < 0.$$

Thus we have

$$Q_2(x)Q_2(1/2 - 4x^2) \ge f(1/4) = (Q_2(1/4))^2, \ \text{for } x \in [\frac{1}{10}, \frac{1}{4}].$$

Assume that $n \geq 3$. Since $(\frac{1}{2}-x)^n \geq (\frac{1}{3}(\frac{1}{2}+x))^n$ for $x \in [\frac{1}{10}, \frac{1}{4}]$, we have

$$\begin{aligned} Q_n(x)Q_n(\frac{1}{2}-4x^2) &= ((\frac{1}{2}-x)^n + (\frac{1}{2}+x)^n)((1-4x^2)^n + (4x^2)^n) \\ &\geq ((\frac{1}{3}(\frac{1}{2}+x))^n + (\frac{1}{2}+x)^n)((1-4x^2)^n + (4x^2)^n) \\ &= ((\frac{1}{3})^n + 1)(\frac{1}{2}+x)^n((1-4x^2)^n + (4x^2)^n). \end{aligned} \tag{1}$$

Let $g_n(x) := (\frac{1}{2}+x)^n \left((1-4x^2)^n + (4x^2)^n\right)$. We claim that

$$g_n(x) \geq \left(\frac{3}{4}\right)^n \left(\left(\frac{3}{4}\right)^n + \left(\frac{1}{4}\right)^n\right), \text{ for } x \in [\frac{1}{10}, \frac{1}{4}]. \tag{2}$$

Indeed, we divide into two cases. Suppose that $x \in [\frac{1}{10}, \frac{1}{5}]$. Then

$$\begin{aligned} (g_n(x))^{1/n} &\geq (\frac{1}{2}+x)(1-4x^2) \geq (\frac{1}{2}+\frac{1}{10})(1-4(\frac{1}{10})^2) = 0.576 \\ &\geq (\frac{3}{4})((\frac{3}{4})^3 + (\frac{1}{4})^3)^{1/3} \approx 0.569. \end{aligned}$$

Noticing that $((\frac{3}{4})^n + (\frac{1}{4})^n)^{1/n}$ is decreasing on n, we obtain Condition (2). Suppose, on the other hand, that $x \in [\frac{1}{5}, \frac{1}{4}]$. We first derive g_n' as follows:

$$\begin{aligned} g_n'(x) &= n(x+\frac{1}{2})^{n-1}\{(1-4x^2)^n + (4x^2)^n\} \\ &+ n(x+\frac{1}{2})^n\{-8x(1-4x^2)^{n-1} + 8x(4x^2)^{n-1}\} \\ &= n(x+\frac{1}{2})^{n-1} \\ &\{(1-4x^2)^n + (4x^2)^n + (x+\frac{1}{2})(-8x(1-4x^2)^{n-1} + 8x(4x^2)^{n-1})\} \\ &= n(x+\frac{1}{2})^{n-1}\{(1-4x^2)^{n-2}((1-4x^2)^2 - 8x(1-4x^2)(x+\frac{1}{2})) \\ &+ (4x^2)^{n-2}((4x^2)^2 + 8x(4x^2)(x+\frac{1}{2}))\}. \end{aligned}$$

Since $4x^2 \leq 1-4x^2$ for $x \in [\frac{1}{5}, \frac{1}{4}]$, we obtain

$$\begin{aligned} g_n'(x) &\leq n(x+\frac{1}{2})^{n-1}\{(1-4x^2)^{n-2}((1-4x^2)^2 - 8x(1-4x^2)(x+\frac{1}{2})) \\ &+ (1-4x^2)^{n-2}((4x^2)^2 + 8x(4x^2)(x+\frac{1}{2}))\} \\ &= n(x+\frac{1}{2})^{n-1}(1-4x^2)^{n-2} \\ &\{(1-4x^2)^2 - 8x(1-4x^2)(x+\frac{1}{2}) + (4x^2)^2 + 8x(4x^2)(x+\frac{1}{2})\} \\ &= n(x+\frac{1}{2})^{n-1}(1-4x^2)^{n-2}(96x^4 + 32x^3 - 16x^2 - 4x + 1). \end{aligned} \tag{3}$$

Let $h(x) := 96x^4 + 32x^3 - 16x^2 - 4x + 1$. Then

$$h'(x) = 384(x-\frac{1}{4})(x - \frac{-3+\sqrt{3}}{12})(x - \frac{-3-\sqrt{3}}{12}).$$

Thus $h'(x) \leq 0$ for $x \in [\frac{1}{5}, \frac{1}{4}]$. Since $h(\frac{1}{5}) = -\frac{19}{625} < 0$, $h(x) < 0$ for $x \in [\frac{1}{5}, \frac{1}{4}]$. This together with (3) imply that $g_n'(x) < 0$ for $x \in [\frac{1}{5}, \frac{1}{4}]$. Hence

$$g_n(x) \geq g_n(\frac{1}{4}) = \left(\frac{3}{4}\right)^n \left(\left(\frac{3}{4}\right)^n + \left(\frac{1}{4}\right)^n\right) \text{ for } x \in [\frac{1}{5}, \frac{1}{4}].$$

This concludes the claim. Putting this back to Condition (1), we obtain that for $x \in [\frac{1}{10}, \frac{1}{4}]$,

$$\begin{aligned} Q_n(x)Q_n(\frac{1}{2} - 4x^2) &\geq ((\frac{1}{3})^n + 1)(\frac{3}{4})^n((\frac{3}{4})^n + (\frac{1}{4})^n) \\ &= ((\frac{3}{4})^n + (\frac{1}{4})^n)^2 \\ &= (Q_n(\frac{1}{4}))^2. \end{aligned}$$

Finally, we check the condition (c). Note that since $Q_1(y) \equiv 1$, the condition (3) is obviously true for $n = 1$. Suppose that $n \geq 2$. It is obvious by elementary calculation that for $x \in [0, \frac{1}{10}]$,

$$\begin{aligned} &Q_n(x) = (\frac{1}{2} - x)^n + (\frac{1}{2} + x)^n \geq (\frac{1}{3})^n(\frac{1}{2} + x)^n + (\frac{1}{2} + x)^n; \\ &Q_n(\frac{1}{2} - 4x^2) = (1 - 4x^2)^n + (4x^2)^n \geq (1 - 4x^2)^n \geq (1 - 4(\frac{1}{10})^2)^n = (\frac{96}{100})^n; \\ &Q_n(-64x^4 + 16x^2 - \frac{1}{2}) = (1 - 8x^2)^{2n} + (16x^2(1 - 4x^2))^n \geq (1 - 8x^2)^{2n}. \end{aligned}$$

These imply that

$$(Q_n(x)Q_n(\frac{1}{2} - 4x^2)Q_n(-64x^4 + 16x^2 - \frac{1}{2}))^{1/n} \geq ((\frac{1}{3})^n + 1)^{1/n}(\frac{1}{2} + x)\frac{96}{100}(1 - 8x^2)^2.$$

Since $(\frac{1}{2} + x)(1 - 8x^2)^2 \geq \frac{1}{2}$ for $x \in [0, \frac{1}{10}]$, we have

$$(\frac{1}{2} + x)\frac{96}{100}(1 - 8x^2)^2 \geq \frac{48}{100} > \frac{15}{32} = \frac{3}{4}((\frac{3}{4})^2 + (\frac{1}{4})^2)^{2/2}.$$

Noticing that $((\frac{3}{4})^n + (\frac{1}{4})^n)^{2/n}$ is decreasing on n, we have

$$\begin{aligned} &(Q_n(x)Q_n(\frac{1}{2} - 4x^2)Q_n(-64x^4 + 16x^2 - \frac{1}{2}))^{1/n} \\ &> ((\frac{1}{3})^n + 1)^{1/n}\frac{3}{4}((\frac{3}{4})^n + (\frac{1}{4})^n)^{2/n} \\ &= ((\frac{3}{4})^n + (\frac{1}{4})^n)^{3/n} \\ &= (Q_n(1/4))^{3/n}. \end{aligned}$$

This completes the proof. □

Theorem 3 *Let φ_n be the Butterworth refinable function with order n. Then*

$$|\hat{\varphi}_n(\xi)| \leq C(1 + |\xi|)^{-2n+\kappa_n}, \tag{4}$$

where $\kappa_n = \log_2(Q_n(\frac{1}{4})) = 2n - n\log_2 3 - \log_2(1 + 3^{-n})$ and this decay is optimal; i.e., $s_\infty(\varphi_n) = 2n - \kappa_n = n\log_2 3 + \log_2(1 + 3^{-n})$. In particular, $|\hat{\varphi}_n(\xi)| \leq C(1 + |\xi|)^{-n\log_2 3}$ and $\varphi_n \in C^s$ for any $s < n\log_2 3 - 1$.

Proof. Recall that

$$b_n(\xi) = \frac{\cos^{2n}(\xi/2)}{\cos^{2n}(\xi/2) + \sin^{2n}(\xi/2)}.$$

Since

$$Q_n(\sin^2(\xi/2) - 1/2) = \cos^{2n}(\xi/2) + \sin^{2n}(\xi/2),$$

$|\mathcal{L}(w)|$ in Proposition 1 is exactly $(Q_n(\sin^2(\xi/2) - 1/2)^{-1}$ here. Let $x := \sin^2(\xi/2) - 1/2$. Then we have

$$\begin{aligned} |\mathcal{L}(2\xi)| &= (Q_n(\sin^2(\xi) - \frac{1}{2}))^{-1} \\ &= (Q_n(4\sin^2(\frac{\xi}{2})(1 - \sin^2(\frac{\xi}{2})) - \frac{1}{2}))^{-1} = (Q_n(\frac{1}{2} - 4x^2))^{-1}. \end{aligned}$$

Similarly,

$$|\mathcal{L}(4\xi)| = \left(Q_n(-64x^4 + 16x^2 - 1/2)\right)^{-1}.$$

We take

$$\begin{aligned}
D_1 &:= [-\pi, -2\pi/3] \cup [2\pi/3, \pi];\\
D_2 &:= [-2\pi/3, -2\sin^{-1}(\sqrt{3/5})] \cup [2\sin^{-1}(\sqrt{3/5}), 2\pi/3];\\
D_3 &:= [-2\sin^{-1}(\sqrt{3/5}), 2\sin^{-1}(\sqrt{3/5})].
\end{aligned}$$

Then it is easy to see that

$$\begin{aligned}
\xi \in D_1 &\Leftrightarrow x \in [-1/2, -1/4] \cup [1/4, 1/2];\\
\xi \in D_2 &\Leftrightarrow x \in [-1/4, -1/10] \cup [1/10, 1/4];\\
\xi \in D_3 &\Leftrightarrow x \in [-1/10, 1/10].
\end{aligned}$$

Hence, by Proposition 1 and Lemma 2, φ satisfies

$$|\hat{\varphi}(\xi)| \le C(1 + |\xi|)^{-2n+\kappa},$$

where $\kappa = \log_2(|\mathcal{L}(2\pi/3)|)$ and this decay is optimal. This leads to $\varphi \in C^s$ for any $s < 2n - \kappa - 1$. □

We can also give the precise Sobolev exponent of a special class of refinable orthonormal cardinal functions from Blaschke products in [2].

Example 4 Consider the rational filter a_n defined by

$$a_n(w) = \frac{(1 + e^{-iw})^{2n+1}}{(1 + e^{-iw})^{2n+1} - (1 - e^{-iw})^{2n+1}},$$

which yields the refinable orthonormal cardinal function φ_n. See [2]. Since

$$|a_n(w)| = \left(\frac{\cos^{2(2n+1)}(w/2)}{\cos^{2(2n+1)}(w/2) + \sin^{2(2n+1)}(w/2)}\right)^{1/2}.$$

Hence, by Theorem 3, we obtain

$$|\hat{\varphi}_n(\xi)| \le C(1 + |\xi|)^{-\frac{1}{2}\{(2n+1)\log_2 3 + \log_2(1+3^{-2n-1})\}}$$

and this decay is optimal; *i.e.*,

$$s_\infty(\varphi_n) = \frac{1}{2}\{(2n+1)\log_2 3 + \log_2(1 + 3^{-2n-1})\}.$$

In particular,

$$|\hat{\varphi}_n(\xi)| \le C(1 + |\xi|)^{-\frac{1}{2}(2n+1)\log_2 3}$$

and

$$\varphi_n \in C^s \text{ for any } s < \frac{1}{2}(2n+1)\log_2 3 - 1.$$

□

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