On asymptotic behavior of Battle-Lemarié scaling functions and wavelets *

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Abstract

We show that the 'centered' Battle-Lemarié scaling function and wavelet of order n converge in $L^q(2 \le q \le \infty)$, uniformly in particular, to the Shannon scaling function and wavelet as n tends to the infinity.

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1 Introduction

The Battle-Lemarié scaling function is obtained by applying the orthogonalization trick to the B-spline functions. In order to get the symmetry about the origin, we will take the centered B-spline of order n as

$$\begin{aligned} B_1(x) &:= \chi_{[-1/2,1/2)}(x), \\ B_n(x) &:= B_{n-1} * B_1(x),\ n = 2, 3, \cdots . \end{aligned} \tag{1.1}$$

The Fourier transform of B_n then has the form

$$\hat{B}_n(w) = \left(\frac{\sin w/2}{w/2}\right)^n = (\cos w/4)^n \hat{B}_n(w/2). \tag{1.2}$$

We note

$$\begin{aligned} \Phi_n(w) &:= \sum_{k\in\mathbb{Z}} |\hat{B}_n(w + 2\pi k)|^2 \\ &= (\cos w/4)^{2n}\Phi_n(w/2) + (\sin w/4)^{2n}\Phi_n(w/2 + \pi). \end{aligned} \tag{1.3}$$

and apply the orthonormalization trick to B_n to get the Battle-Lemarié scaling function φ_n of order n defined by

$$\hat{\varphi}_n(w) := \frac{\hat{B}_n(w)}{\sqrt{\Phi_n(w)}} = m_n(w/2)\hat{\varphi}_n(w/2), \tag{1.4}$$

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where

$$m_n(w) = (\cos w/2)^n \sqrt{\frac{\Phi_n(w)}{\Phi_n(2w)}}. \tag{1.5}$$

The filter m_n is 2π-periodic if n is even and 4π-periodic if n is odd. We note that m_n is a CQF filter in the sense that

$$|m_n(w)|^2 + |m_n(w+\pi)|^2 = 1. \tag{1.6}$$

The corresponding wavelet is given by

$$\hat{\psi}_n(2w) = e^{-iw} M_n(w) \hat{\varphi}_n(w), \tag{1.7}$$

where

$$M_n(w) = |(\sin w/2)|^n \sqrt{\frac{\Phi_n(w+\pi)}{\Phi_n(2w)}} = |m_n(w+\pi)|. \tag{1.8}$$

Note that M_n is 2π-periodic. Therefore, if n is even, the function φ_n defines an orthonormal scaling function for a multiresolution analysis. If n is odd, φ_n does not define a scaling function of a multiresolution analysis, but they have the same asymptotic behavior as will be seen in the main theorem in this article. See [1, 2, 3] for the standard Battle-Lemarié wavelet. In this short article, we show that the Battle-Lemarié scaling function φ_n and its corresponding wavelet ψ_n converge, in $L^q(\mathbb{R})(2 \le q \le \infty)$, uniformly in particular, to the Shannon scaling function φ_{SH} and Shannon wavelet ψ_{SH} as n tends to the infinity, where

$$\hat{\varphi}_{SH}(w) := \chi_{[-\pi,\pi]}(w)$$

and

$$\hat{\psi}_{SH}(w) := e^{-iw/2} \chi_{[-2\pi,-\pi]\bigcup[\pi,2\pi]}(w).$$

It is known that the centered B-spline B_n tends to the Gaussian distribution as $n \to \infty$ [4, 5]. For the asymptotic behavior of Daubechies filters and scaling functions, see [6, 7, 8]. The idea of the proof also appears in [9, 10] for the analogous asymptotic behaviors of other family of wavelets.

2 Main result

We need the following property of the Euler-Frobenius polynomials.

Proposition 2.1 ([11]) *Let n be any positive integer and let E_{2n-1} be the Euler-Frobenius polynomial of degree $2n-2$ defined by*

$$E_{2n-1}(z) := (2n-1)! \sum_{k=0}^{2n-2} B_{2n}(-n+k+1) z^k.$$

Then the $2n-2$ roots, $\{\lambda_{n,j} : j = 1, \cdots, 2n-2\}$, of E_{2n-1} has the properties that

$$\lambda_{n,2n-2} < \lambda_{n,2n-3} \cdots < \lambda_{n,n} < -1 < \lambda_{n,n-1} < \cdots < \lambda_{n,1} < 0;$$

$$\lambda_{n,j}\lambda_{n,2n-1-j} = 1, \ (j = 1, 2, \cdots, n-1)$$

and

$$\Phi_n(w) = \frac{e^{iw(n-1)}}{(2n-1)!} E_{2n-1}(e^{-iw}) = \frac{1}{(2n-1)!} \prod_{k=1}^{n-1} \frac{1 - 2\lambda_{n,k}\cos w + \lambda_{n,k}^2}{|\lambda_{n,k}|}.$$

Therefore, $\Phi_n(w+\pi) \le \Phi_n(w)$ on $[-\pi/2, \pi/2]$ and $\Phi_n(w) \le \Phi_n(w+\pi)$ on $[-\pi, -\pi/2) \bigcup (\pi/2, \pi]$.

The 2π-periodic filters for the Shannon scaling function and wavelet are given, respectively, as

$$m^0_{SH}(w) := \begin{cases} 1, & |w| \le \pi/2; \\ 0, & \pi/2 < |w| \le \pi, \end{cases} \tag{2.1}$$

and

$$m^H_{SH}(w) := \begin{cases} 0, & |w| < \pi/2; \\ 1, & \pi/2 \le |w| \le \pi. \end{cases} \tag{2.2}$$

We also define a 4π-periodic filter $m^1_{SH} \in L^2([-2\pi, 2\pi])$ by

$$m^1_{SH}(w) := \begin{cases} 1, & |w| \le \pi/2; \\ 0, & \pi/2 < |w| \le \pi; \\ -1, & \pi < |w| < 3\pi/2; \\ 0, & 3\pi/2 \le |w| \le 2\pi. \end{cases} \tag{2.3}$$

Notice that

$$\hat{\varphi}_{SH}(w) := \chi_{[-\pi,\pi]}(w) = \prod_{j=1}^{\infty} m^0_{SH}(w/2^j) = \prod_{j=1}^{\infty} m^1_{SH}(w/2^j). \tag{2.4}$$

Lemma 2.2 *As n tends to ∞,*

(a) *$m_{2n}(w)$ converges to $m^0_{SH}(w)$ for every $w \in [-\pi, \pi] \setminus \{\pm\pi/2\}$;*

(b) *$m_{2n+1}(w)$ converges to $m^1_{SH}(w)$ for every $w \in [-2\pi, 2\pi] \setminus \{\pm\pi/2, \pm 3\pi/2\}$;*

and so, $M_n(w)$ converges to $m^H_{SH}(w)$ for every $w \in [-\pi, \pi] \setminus \{\pm\pi/2\}$.

Proof. For $w \in (-3\pi/2, -\pi/2) \bigcup (\pi/2, 3\pi/2)$, $\Phi_n(w) \le \Phi_n(w + \pi)$ by Proposition 2.1. By use of (1.3), we see that

$$\begin{aligned} |m_n(w)|^2 &= \frac{(\cos w/2)^{2n}\Phi_n(w)}{\Phi_n(2w)} \\ &= \frac{(\cos w/2)^{2n}}{(\sin w/2)^{2n}} \frac{(\sin w/2)^{2n}\Phi_n(w)}{(\cos w/2)^{2n}\Phi_n(w) + (\sin w/2)^{2n}\Phi_n(w+\pi)} \\ &\le \frac{1}{(\tan w/2)^{2n}} \frac{(\sin w/2)^{2n}\Phi_n(w)}{(\sin w/2)^{2n}\Phi_n(w+\pi)} \\ &\le \frac{1}{(\tan w/2)^{2n}} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Now, let $w \in (-2\pi, -3\pi/2) \bigcup (-\pi/2, \pi/2) \bigcup (3\pi/2, 2\pi)$. Note that

$$|m_n(w)|^2 + |m_n(w + \pi)|^2 = 1. \tag{2.5}$$

Hence $\lim_{n\to\infty} |m_n(w)| = 1$, since $\lim_{n\to\infty} |m_n(w + \pi)| = 0$ by (2.5). Since $m_{2n}(w)$ is 2π-periodic and positive by the definition of m_{2n}, $\lim_{n\to\infty} m_{2n}(w) = 1$. Therefore, (a) is satisfied. For (b), note that m_{2n+1} is 4π-periodic. If $w \in (-\pi/2, \pi/2)$, then $m_{2n+1}(w)$ is positive. Hence $\lim_{n\to\infty} m_{2n+1}(w) = 1$. If $w \in (-2\pi, -3\pi/2) \bigcup (3\pi/2, 2\pi)$, then $m_{2n+1}(w)$ is negative. Therefore $\lim_{n\to\infty} m_{2n+1}(w) = -1$. □

Lemma 2.3 *For all n,*

$$|m_n(w) - 1| \le \begin{cases} 2, & \textit{for all } w; \\ 2|w|/\pi, & |w| \le \pi/2. \end{cases}$$

Proof. We note that

$$|m_n(w) - 1| \leq |m_n(w)| + 1 \leq 2.$$

For $|w| \leq \pi/2$ and for $n \geq 1$, $|\tan w/2|^{2n} \leq |\tan w/2| \leq 2|w|/\pi$. Therefore, we have for $|w| \leq \pi/2$,

$$\begin{aligned}
|m_n(w) - 1| &= \left| \sqrt{\frac{\Phi_n(w)}{\Phi_n(2w)}}(\cos w/2)^n - 1 \right| \\
&= \left| \frac{\sqrt{\Phi_n(w)}(\cos w/2)^n - \sqrt{\Phi_n(2w)}}{\sqrt{\Phi_n(2w)}} \right| \\
&= \left| \frac{\Phi_n(w)(\cos w/2)^{2n} - \Phi_n(2w)}{\sqrt{\Phi_n(2w)}(\sqrt{\Phi_n(w)}(\cos w/2)^n + \sqrt{\Phi_n(2w)})} \right| \\
&\leq \frac{(\sin w/2)^{2n}\Phi_n(w+\pi)}{\Phi_n(2w)} \\
&= \frac{(\sin w/2)^{2n}}{(\cos w/2)^{2n}} \frac{(\cos w/2)^{2n}\Phi_n(w+\pi)}{\Phi_n(2w)} \\
&= (\tan w/2)^{2n} \frac{(\cos w/2)^{2n}\Phi_n(w+\pi)}{(\cos w/2)^{2n}\Phi_n(w) + (\sin w/2)^{2n}\Phi_n(w+\pi)} \\
&\leq (\tan w/2)^{2n} \frac{\Phi_n(w+\pi)}{\Phi_n(w)} \\
&\leq \frac{2}{\pi}|w|,
\end{aligned}$$

where we used the fact that $\Phi_n(w+\pi) \leq \Phi_n(w)$ on $[-\pi/2, \pi/2]$. □

Lemma 2.4 (a) *For each fixed w, $\hat{\varphi}_n(w) = \prod_{j=1}^{\infty} m_n(w/2^j)$ converges uniformly on n.*

(b) *$\hat{\varphi}_n(w) \to \hat{\varphi}_{SH}(w)$ pointwise a.e. as $n \to \infty$.*

(c) *$\hat{\psi}_n(w) \to \hat{\psi}_{SH}(w)$ pointwise a.e. as $n \to \infty$.*

Proof. (a) Fix w and choose j_0 so that $|w/2^{j_0}| \leq \pi/2$. By Lemma 2.3,

$$\begin{aligned}
\sum_{j=1}^{\infty} |m_n(\frac{w}{2^j}) - 1| &= \sum_{j=1}^{j_0} |m_n(\frac{w}{2^j}) - 1| + \sum_{j=j_0+1}^{\infty} |m_n(\frac{w}{2^j}) - 1| \\
&\leq 2j_0 + \sum_{j=j_0+1}^{\infty} \frac{2}{\pi}\frac{|w|}{2^j} = 2j_0 + \frac{2}{\pi}\frac{|w|}{2^{j_0}},
\end{aligned}$$

uniformly on n. Therefore, the product $\varphi_n(w)$ converges uniformly on n.
(b) Fix $w \notin \cup_{j=1}^{\infty} 2^j(\pm\pi + 2\pi\mathbb{Z})$ and let $\epsilon > 0$. By (a), we can choose j_1 (independent of n) so that

$$|\hat{\varphi}_n(w) - \prod_{j=1}^{j_1} m_n(\frac{w}{2^j})| < \epsilon,$$

and

$$|\hat{\varphi}_{SH}(w) - \prod_{j=1}^{j_1} m_{SH}^i(\frac{w}{2^j})| < \epsilon,$$

for $i = 0, 1$. Therefore, we have

$$|\hat{\varphi}_n(w) - \hat{\varphi}_{SH}(w)| \le |\hat{\varphi}_n(w) - \prod_{j=1}^{j_1} m_n(\frac{w}{2^j})| + |\prod_{j=1}^{j_1} m_n(\frac{w}{2^j}) - \prod_{j=1}^{j_1} m^i_{SH}(\frac{w}{2^j})| + |\prod_{j=1}^{j_1} m^i_{SH}(\frac{w}{2^j}) - \hat{\varphi}_{SH}(w)|$$
$$< 2\epsilon + |\prod_{j=1}^{j_1} m_n(\frac{w}{2^j}) - \prod_{j=1}^{j_1} m^i_{SH}(\frac{w}{2^j})|.$$

We choose $i := i(n) = 0$ (n=even), 1 (n=odd). Note that $w/2^j \notin \pm\pi/2 + 2\pi\mathbb{Z}$ for any $j \ge 1$. Since $m_{2n}(w/2^j) \to m^0_{SH}(w/2^j)$ and $m_{2n+1}(w/2^j) \to m^1_{SH}(w/2^j)$ as $n \to \infty$ as shown in Lemma 2.2, we can choose $n_0 \in \mathbb{N}$ so that

$$|\prod_{j=1}^{j_1} m_n(w/2^j) - \prod_{j=1}^{j_1} m^{i(n)}_{SH}(w/2^j)| < \epsilon \quad \text{for } n \ge n_0.$$

Therefore, $\hat{\varphi}_n(w) \to \hat{\varphi}_{SH}(w)$ pointwise as $n \to \infty$ for $w \notin \cup_{j=1}^{\infty} 2^j(\pm\pi + 2\pi\mathbb{Z})$.
(c) The proof follows from (b) in view of the definition of $\hat{\psi}_n$ in (1.7). It is also proved in [3] with a different proof.

□

Now, we state and prove our main result.

Theorem 2.5 (a) *For* $1 \le p < \infty$, $||\hat{\varphi}_n - \hat{\varphi}_{SH}||_{L^p(\mathbb{R})} \to 0$ *and* $||\hat{\psi}_n - \hat{\psi}_{SH}||_{L^p(\mathbb{R})} \to 0$ *as* $n \to \infty$.
(b) *For* $2 \le q \le \infty$, $||\varphi_n - \varphi_{SH}||_{L^q(\mathbb{R})} \to 0$ *and* $||\psi_n - \psi_{SH}||_{L^q(\mathbb{R})} \to 0$, *as* $n \to \infty$.
In particular, $\varphi_n \to \varphi_{SH}$ *and* $\psi_n \to \psi_{SH}$ *uniformly on* $\mathbb{R}$ *as* $n \to \infty$.

Proof. We define an auxiliary 2π-periodic continuous function M, via

$$M(w) = \begin{cases} 1, & |w| \le \frac{\pi}{2}; \\ 2^{3/2}(\cos w/2)^3, & \frac{\pi}{2} < |w| \le \pi, \end{cases}$$

and let $\hat{\varphi}(w) := \prod_{j=1}^{\infty} M(w/2^j)$. It is obvious that

$$0 \le |m_n(w)| \le M(w), \ n = 3, 4, \cdots$$

and that $\hat{\varphi}(w)$ has the decay $|\hat{\varphi}(w)| \le C(1 + |w|)^{-3/2}$ by Theorem 5.5 of [11]. We have

$$|\hat{\varphi}_n(w)| = \prod_{j=1}^{\infty} |m_n(\frac{w}{2^j})|$$
$$\le \prod_{j=1}^{\infty} |M(\frac{w}{2^j})| = |\hat{\varphi}(w)| \le C(1 + |w|)^{-3/2},$$
$$|\hat{\psi}_n(w)| = M_n(w/2)|\hat{\varphi}_n(w/2)| \le C(1 + |w/2|)^{-3/2}.$$

Therefore (a) follows from Lemma 2.4 by the dominated convergence theorem. (b) follows from (a) by Hausdorff-Young inequality:

$$||f||_{L^q(\mathbb{R})} \le ||\hat{f}||_{L^p(\mathbb{R})}, \ \text{for } 1 \le p \le 2,$$

where q is the conjugate exponent to p. □

Remark. We illustrate the convergence of the Battle-Lemarié scaling functions and wavelets (for $n = 4$ and 10) to the Shannon scaling function and wavelet in Figure 1.

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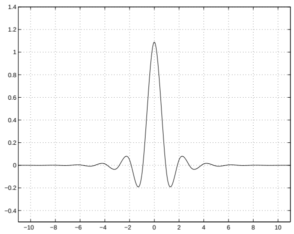
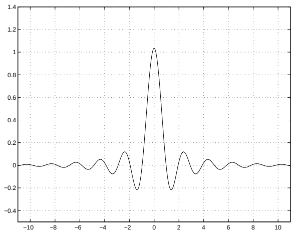


(a)



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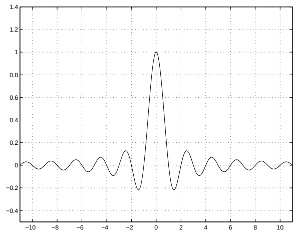

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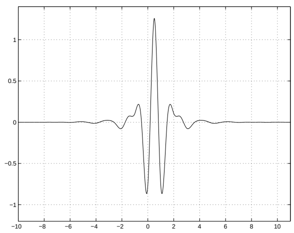

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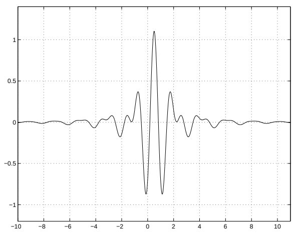

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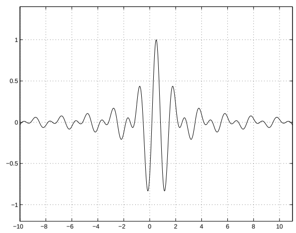

(f)

Figure 1: (a) φ_4 (b) φ_{10} (c) φ_{SH} (d) ψ_4 (e) ψ_{10} (f) ψ_{SH}.

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