Riesz sequences of translates and generalized duals with support on $[0, 1]$.

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Abstract

If the integer translates of a function ϕ with compact support generate a frame for a subspace $\mathcal{W}$ of $L^2(\mathbb{R})$, then it is automatically a Riesz basis for $\mathcal{W}$, and there exists a unique dual Riesz basis belonging to $\mathcal{W}$. Considerable freedom can be obtained by considering oblique duals, i.e., duals not necessarily belonging to $\mathcal{W}$. Extending work by Ben–Artzi and Ron, we characterize the existence of oblique duals generated by a function with support on an interval of length one. If such a generator exists, we show that it can be chosen with desired smoothness. Regardless whether ϕ is polynomial or not, the same condition implies that a polynomial dual supported on an interval of length one exists.

1 Introduction

Let $L^2(\mathbb{R})$ denote the *real* Hilbert space consisting of real-valued square-integrable functions on $\mathbb{R}$, and with the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx.$$

Let T_k, $k \in \mathbb{Z}$, denote the translation operator $(T_k f)(x) = f(x-k), x \in \mathbb{R}$. In this paper we aim at construction of two functions $\phi, \tilde{\phi} \in L^2(\mathbb{R})$ with

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compact support, for which the expansions

$$f = \sum_{k\in\mathbb{Z}} \langle f, T_k\tilde{\phi}\rangle T_k\phi, \ f \in \overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}} \tag{1}$$

hold. Our goal is to obtain expansions which can be implemented directly, without modification of the functions ϕ and $\tilde{\phi}$; this means that we want the functions ϕ and $\tilde{\phi}$ to have compact support and be given explicitly in terms of (finite linear combinations of) elementary functions. Very often we also want ϕ and $\tilde{\phi}$ to be differentiable up to a certain order.

In our approach, we fix a function ϕ having compact support and for which $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz basis for its closed linear span, and search for a nice (in the above sense) function $\tilde{\phi}$ such that the expansion property (1) holds. We note that the generator for the canonical dual frame of $\{T_k\phi\}_{k\in\mathbb{Z}}$ in general does not satisfy our requirements: in general, this function is an *infinite* linear combination of the functions $T_k\phi, \ k \in \mathbb{Z}$, and does not have compact support. The generator for the canonical dual frame belongs to $\mathcal{W} := \overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}}$. We will see that, by allowing $\tilde{\phi}$ to be *outside* $\mathcal{W}$, we obtain considerable freedom, which, under a natural condition, allows us to construct functions $\tilde{\phi}$ with the properties we want.

We note that results of that type are known for B-splines, due to work by deBoor [3], [4]. Furthermore, in 1990 Ben–Artzi and Ron [1] obtained results in the spirt of what we are aiming at. In particular, they proved the following:

Theorem 1.1 *Assume that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence and that ϕ has support in $[0, N]$ for some $N \in \mathbb{N}$. Then the following holds:*

(i) The expansion property (1) *holds with a compactly supported function $\tilde{\phi}$ if and only if the only solution to the equation $\sum_{k\in\mathbb{Z}} c_kT_k\phi = 0$ is $c_k = 0, \ \forall k$;*

(ii) If the functions $x \mapsto \phi(x), x \mapsto \phi(x + 1), \dots, x \mapsto \phi(x + N - 1)$ are linearly independent, there exists a function $\tilde{\phi}$ with support in $[0, 1]$ *such that* (1) *holds.*

One of our results, Corollary 3.2, characterizes the independence condition in Theorem 1.1 (ii).

A few remarks about terminology are in order. Considering a function $\phi \in L^2(\mathbb{R})$ such that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence, the unique function $\tilde{\phi} \in \mathcal{W}$ satisfying (1) is called the (*canonical*) dual generator. *Any* functiom $\tilde{\phi} \in L^2(\mathbb{R})$ satisfying (1) is called a *generalized dual generator*; and if a generalized dual generator $\tilde{\phi}$ has the additional property that $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ is a frame sequence, then $\tilde{\phi}$ is called an *oblique dual generator*. This terminology is used, e.g., in [6]. All our constructions yield oblique dual generators supported on an interval of the form $[n, n+1]$ for some $n \in \mathbb{Z}$; thus, $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ is in fact an orthogonal system. We further note that while the expansions we obtain in (1) are more general than classical frame decompositions, they are a special case of the pseudo-frame decompositions considered by Li and Ogawa [9]. Oblique duals of Riesz bases of translates played a key role in the construction of a biorthogonal multiresolution analysis in [8].

Note that [6] also deals with frames of translates and their oblique duals, but with a different setup: the results in [6] yield theoretical conditions for the existence of oblique duals belonging to precribed vector spaces, but only few concrete constructions.

2 Basic results

Let $\mathcal{H}$ denote a separable *real* Hilbert space with inner product $\langle\cdot,\cdot\rangle$. We assume that the reader is familiar with the concepts of frames, Riesz bases, and Bessel sequences in Hilbert spaces.

Any Riesz basis $\{f_k\}_{k=1}^\infty$ for $\mathcal{H}$ is also a frame for $\mathcal{H}$. On the other hand, a frame $\{f_k\}_{k=1}^\infty$ is a Riesz basis if

$$\sum_{k=1}^\infty c_k f_k = 0,\ \{c_k\} \in \ell^2 \Rightarrow c_k = 0,\ \forall k.$$

A sequence $\{f_k\}_{k=1}^\infty$ in $\mathcal{H}$ which only forms a frame (resp. Riesz basis) for a closed subspace of $\mathcal{H}$ is called a *frame sequence* (resp. *Riesz sequence*).

It is well–known that if $\phi \in L^2(\mathbb{R})$ has compact support and $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence, then it is automatically a Riesz sequence; a proof can be found, e.g., in [5]. The following lemma will be the starting point for our constructions.

Lemma 2.1 *Assume that the functions $\phi, \tilde{\phi} \in L^2(\mathbb{R})$ have compact support and generate Bessel sequences. Then the following are equivalent:*

(i) $f = \sum_{k\in\mathbb{Z}} \langle f, T_k\tilde{\phi}\rangle T_k\phi,\ \forall f \in \mathcal{W}$*;*

(ii) $\langle \phi, T_k\tilde{\phi}\rangle = \delta_{k,0}$.

Proof. If (i) holds, then $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{W}$, and therefore a Riesz sequence; (ii) follows by letting $f = \phi$ in (i). On the other hand, if (ii) holds, then (i) holds for $f = \phi$. A change of the summation index proves that then (i) holds for any translate $T_k\phi$, and therefore on span$\{T_k\phi\}_{k\in\mathbb{Z}}$; by continuity of the operator $f \mapsto \sum_{k\in\mathbb{Z}}\langle f, T_k\tilde{\phi}\rangle T_k\phi$ we obtain the conclusion. $\square$

Assume now that we fix the function ϕ and want to find a function $\tilde{\phi}$ such that the expansion property (1) holds. If we want to obtain (1) via Lemma 2.1, then we will have to solve the equations in (ii). Our main tool will be to reformulate these equations as a *moment problem*. We refer to the appendix for more information on moment problems.

Lemma 2.2 *Let $\{f_k\}_{k=1}^N$ be a collection of vectors in $\mathcal{H}$ and consider the moment problem*

$$\langle f, f_k\rangle = \begin{cases} 1 \text{ if } k = 1, \\ 0 \text{ if } k = 2, ..., N. \end{cases} \tag{2}$$

Then the following are equivalent:

(i) The moment problem (2) *has a solution f.*

(ii) If $\sum_{k=1}^N c_k f_k = 0$ for some scalar coefficients c_k, then $c_1 = 0$.

(iii) $f_1 \notin span\{f_k\}_{k=2}^N$.

In case a solution exists, it can be chosen of the form $f = \sum_{k=1}^N d_k f_k$ for some scalar coefficients d_k.

The proof is given in the appendix. For our purpose, it turns out to be essential to find other solutions to the moment problem (2) than the one stated in Lemma 2.2. In the next theorem we find solutions belonging to other vector spaces.

Theorem 2.3 *Assume that the vectors $\{f_k\}_{k=1}^N$ in $\mathcal{H}$ satisfy the condition (ii) in Lemma 2.2 and that $\{g_j\}_{j=0}^\infty$ is total in $\mathcal{H}$. Then there exists $M \in \mathbb{N}$ such that the moment problem* (2) *has a solution $f \in span\{g_j\}_{j=0}^M$.*

The proof of Theorem 2.3 is given in the appendix.

For our examples, we will mainly consider B-splines. Recall that they are given inductively by

$$B_1(x) = \chi_{[0,1]},\ B_{N+1}(x) = (B_N * B_1)(x) = \int_0^1 B_N(x-t)dt.$$

3 Oblique dual generators supported on $[0,1]$

Given a function $\phi \in L^2(\mathbb{R})$ with compact support such that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence, our first goal is to search for oblique dual generators supported on an interval of length one. For convenience, we first consider the interval $[0,1]$.

By translation, we can always assume that the generator ϕ for the Riesz basis $\{T_k\phi\}_{k\in\mathbb{Z}}$ has support in an interval $[0,N]$ for some $N \in \mathbb{N}$. Note that we do not require the support to *equal* such an interval.

Theorem 3.1 *Assume that $\phi \in L^2(\mathbb{R})$ has support in an interval $[0,N]$ for some $N \in \mathbb{N}$ and that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence. Then the following are equivalent:*

(i) $\{T_k\phi\}_{k\in\mathbb{Z}}$ *has a generalized dual* $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ *for which* $\operatorname{supp}\tilde{\phi} \subseteq [0,1]$.

(ii) If $\sum_{k=0}^{N-1} c_k\phi(x+k) = 0$ *for all* $x \in [0,1]$, *then* $c_0 = 0$.

(iii) $\phi|_{[0,1]} \notin \operatorname{span}\{(T_{-N+1}\phi)|_{[0,1]}, \cdots, (T_{-N+2}\phi)|_{[0,1]}, \cdots, (T_{-1}\phi)|_{[0,1]}\}$.

In case the conditions are satisfied, the generalized duals $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ are orthogonal sequences; in particular, they are oblique duals of $\{T_k\phi\}_{k\in\mathbb{Z}}$. One can choose $\tilde{\phi}$ of the form

$$\tilde{\phi}(x) = \left(\sum_{k=0}^{N-1} d_k\phi(x+k)\right)\chi_{[0,1]}(x) \tag{3}$$

for some scalar coefficients d_k.

Proof. We will apply Lemma 2.1. If $\operatorname{supp}\tilde{\phi} \subseteq [0,1]$, then

$$\langle T_k\phi, \tilde{\phi}\rangle = \int_0^1 \phi(x-k)\tilde{\phi}(x)dx.$$

Since we have assumed that $\operatorname{supp}\phi \subseteq [0,N]$, this expression shows that for all $k > 0$ and all $k \le -N$, we have $\langle T_k\phi, \tilde{\phi}\rangle = 0$. Thus, the duality condition in Lemma 2.1 means that

$$\begin{cases} 1 = \langle \phi, \tilde{\phi}\rangle = \displaystyle\int_0^1 \phi(x)\tilde{\phi}(x)dx \\ 0 = \langle T_{-1}\phi, \tilde{\phi}\rangle = \displaystyle\int_0^1 \phi(x+1)\tilde{\phi}(x)dx \\ 0 = \cdots\cdots \\ 0 = \langle T_{-N+1}\phi, \tilde{\phi}\rangle = \displaystyle\int_0^1 \phi(x+N-1)\tilde{\phi}(x)dx \end{cases} \tag{4}$$

This is a moment problem in the Hilbert space $L^2(0,1)$, and the stated equivalence follows immediately from Lemma 2.2. Furthermore, if the condition (ii) is satisfied, it is clear that $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ is an orthogonal sequence. $\square$

Note that the coefficients d_k in (3) are easy to find: inserting this expression for $\tilde{\phi}$ into the moment problem (4) leads to a system of N linear equations in the coefficients, $d_0, d_1, \dots, d_{N-1}$.

It is clear that the condition in Theorem 3.1 (ii) is weaker than the condition in Theorem 1.1 (ii): for example, if $g = 0$ on an interval $[k, k+1]$ for some $k = 1, 2, \dots, N-1$, then the condition in Theorem 3.1 (ii) is not satisfied, but Theorem 3.1 (ii) might be satisfied. Further, the reader can check that the function $\phi(x) = x\chi_{[0,1]}(x) + \chi_{[2,3]}(x)$ generates a Riesz sequence, and that it satisfies the condition in Theorem 3.1 (ii); however, the condition in Theorem 1.1 (ii) is not satisfied.

Exactly the same argument as in the proof of Theorem 3.1 gives a necessary and sufficient condition for the existence of an oblique dual generated by a function with support on an interval $[n, n+1]$ for some $n = 0, 1, \dots, N-1$. In fact, under the assumptions in Theorem 3.1, the following are equivalent:

(i) $\{T_k\phi\}_{k\in\mathbb{Z}}$ has an oblique dual $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ for which $\text{supp}\tilde{\phi} \subseteq [n, n+1]$.

(ii) If $\sum_{k=0}^{N-1} c_k\phi(x+k) = 0$ for all $x \in [0,1]$, then $c_n = 0$.

An immediate consequence of this is that the independence condition in Theorem 1.1 (ii) *characterizes* the case where for each $n = 0, 1, \dots, N-1$ there exist oblique dual generators supported on $[n, n+1]$:

Corollary 3.2 *Assume that $\phi \in L^2(\mathbb{R})$ has support on an interval $[0, N]$ and that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence. Then the following are equivalent:*

(i) The functions

$$x \mapsto \phi(x), x \mapsto \phi(x+1), \dots, x \mapsto \phi(x+N-1)$$

are linearly independent on the interval $[0,1]$;

(ii) For each $n = 0, 1, \dots, N-1$, there exists an oblique dual generated by a function with support on $[n, n+1]$.

It is known that for any B-spline B_N, the functions $B_N(\cdot + k)$, $k = 0, \cdots, N-1$, are linearly independent on $[0,1]$. Thus, for each $n = 0, 1, \dots, N-1$, the Riesz sequence $\{T_kB_N\}_{k\in\mathbb{Z}}$ has an oblique dual generator of the form

$$\tilde{\phi}(x) = \left(\sum_{k=0}^{N-1} d_k B_N(x+k)\right)\chi_{[n,n+1]}(x). \tag{5}$$

Example 3.3 Assume that ϕ satisfies the conditions in Theorem 3.1. By Theorem 2.3 it immediately follows that $\{T_k\phi\}_{k\in\mathbb{Z}}$ has an oblique dual generated by a trigonometric polynomial

$$\tilde{\phi}(x) = \left(a_0 + \sum_{n=1}^{M}(a_n \cos(2\pi nx) + b_n \sin(2\pi nx))\right)\chi_{[0,1]}(x).$$

For completeness, we mention that there actually exist functions ϕ with compact support for which $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence, but for which no compactly supported dual generator exists. Consider, e.g., the function

$$\phi = \chi_{[0,1]} + \frac{1}{2}\chi_{[1,2]}$$

A direct verification (or a perturbation argument, see, e.g., Example 15.1.2 in [5]) reveals that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence. Via Theorem 1.1 (i) it is easy to see that no compactly supported generalized dual generator exists.

4 Smooth oblique dual generators

For notational convenience, we stick to oblique dual generators supported on $[0,1]$ in this section. Let us look once more at the expression for the oblique dual generator $\tilde{\phi}$ in (3). We observe that even if the generator ϕ is smooth, $\tilde{\phi}$ will in general not be differentiable at $x = 0$ and $x = 1$. In that case a smoother dual generator can be obtained via the following result. Note that no extra assumption (compared to Theorem 3.1) is needed:

Theorem 4.1 *Assume that $\phi \in L^2(\mathbb{R})$ has support on an interval $[0,N]$ and that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence. Assume that*

$$\sum_{k=0}^{N-1} c_k\phi(x+k) = 0, \forall x \in [0,1] \Rightarrow c_0 = 0.$$

Then, for any $p, q \in \mathbb{N}$, there exists an oblique dual generator of the form

$$\tilde{\phi}(x) = x^p(1-x)^q \left(\sum_{k=0}^{N-1} d_k\phi(x+k)\right)\chi_{[0,1]}(x). \tag{6}$$

Proof. Due to the assumption, we know that if

$$\sum_{k=0}^{N-1} c_k\phi(x+k)x^{p/2}(1-x)^{q/2} = 0 \text{ for all } x \in [0,1],$$

then $c_0 = 0$. Thus, according to Lemma 2.2 with $\mathcal{H} = L^2(0,1)$, the moment problem

$$\begin{cases} 1 = \int_0^1 \phi(x) x^{p/2}(1-x)^{q/2} h(x)dx \\ 0 = \int_0^1 \phi(x+1) x^{p/2}(1-x)^{q/2} h(x)dx \\ 0 = \cdots\cdots \\ 0 = \int_0^1 \phi(x+N-1) x^{p/2}(1-x)^{q/2} h(x)dx \end{cases}$$

has a solution h of the form

$$h(x) = \left(\sum_{k=0}^{N-1} d_k \phi(x+k) x^{p/2}(1-x)^{q/2} \right) \chi_{[0,1]}(x).$$

This means that the function

$$\tilde{\phi}(x) := x^{p/2}(1-x)^{q/2} h(x) = \left(\sum_{k=0}^{N-1} d_k \phi(x+k) \right) x^p (1-x)^q \chi_{[0,1]}(x)$$

solves the moment problem (4). $\square$

We note that in order to apply Theorem 4.1 we do not need to worry about its proof: we may simply take a function $\tilde{\phi}$ of the form (6) and determine the constants $d_0, \dots, d_{N-1}$ such that we obtain a solution to the moment problem (4). On matrix form, the constants $d_0, \dots, d_{N-1}$ are determined by the equation

$$M\mathbf{d} = \mathbf{e},$$

where M is the symmetric matrix

$$\begin{pmatrix} \int_0^1 x^p(1-x)^q \phi(x)\phi(x)dx & \cdot & \cdot & \int_0^1 x^p(1-x)^q \phi(x)\phi(x+N-1)dx \\ \int_0^1 x^p(1-x)^q \phi(x+1)\phi(x)dx & \cdot & \cdot & \int_0^1 x^p(1-x)^q \phi(x+1)\phi(x+N-1)dx \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int_0^1 x^p(1-x)^q \phi(x+N-1)\phi(x)dx & \cdot & \cdot & \int_0^1 x^p(1-x)^q \phi(x+N-1)\phi(x+N-1)dx \end{pmatrix}$$

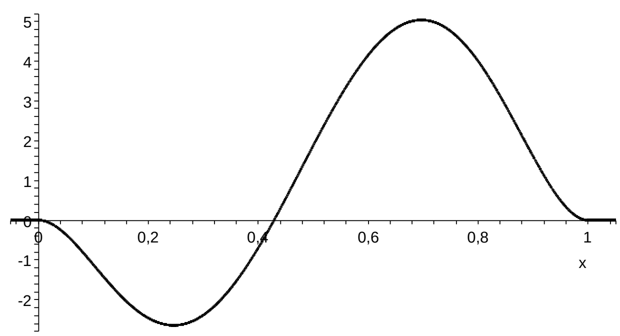
and

$$\mathbf{d} = \begin{pmatrix} d_0 \\ d_1 \\ \cdot \\ \cdot \\ d_{N-1} \end{pmatrix}, \ \mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

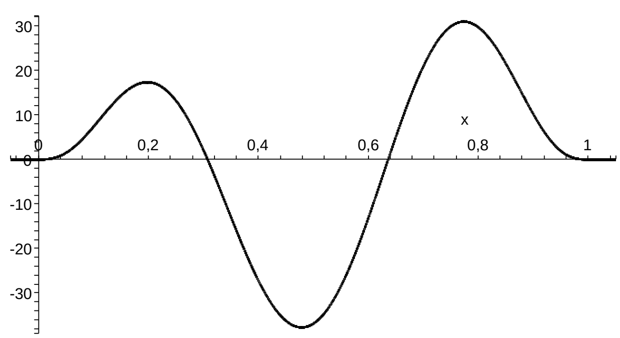
We note that if we want higher order derivatives of $\tilde{\phi}$ at the points $x = 0$ and $x = 1$ to exist, it only affects the integrals in the entries of matrix M, but not the size of the matrix. Thus, the computational complexity does not increase.

Example 4.2 The figures show some oblique dual generators for Riesz sequences generated by various B-splines, for various values of p and q. Note that the existence of smooth oblique dual generators is known from the literature in this special case.

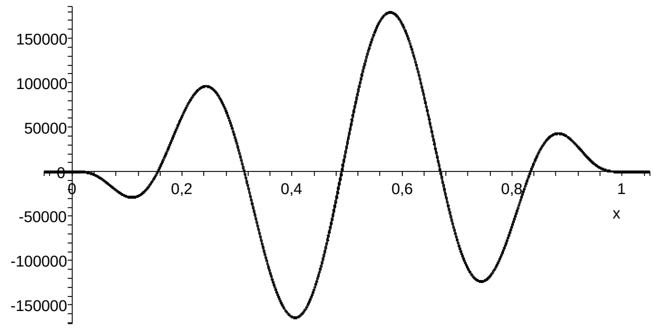
Observe that the oblique dual generator associated with $\phi = B_6, p = q = 4$ is a polynomial of degree 13 on the interval $[0, 1]$ and that it oscillates heavily!



The generator $\tilde{\phi}$ in (6) corresponding to $\phi = B_2,\ p = q = 2$.



The generator $\tilde{\phi}$ in (6) corresponding to $\phi = B_3,\ p = q = 3$.



The generator $\tilde{\phi}$ in (6) corresponding to $\phi = B_6,\ p = q = 4$.

5 Polynomial oblique dual generators

The "smoothening procedure" described in Section 4 only works if the generator ϕ has derivatives of sufficiently high order (except maybe at points $x \in \mathbb{Z}$). For example, if ϕ is non–differentiable at $x = 1/2$, the dual in Theorem 4.1 will in general not be differentiable, regardless how p, q are chosen. Our aim now is to prove that if the necessary condition in Theorem 3.1 is satisfied, then a polynomial dual with support on $[0,1]$ exists, regardless whether ϕ is a polynomial or not. As a corollary of that, we will be able to find smooth oblique generators via the same procedure as before.

Theorem 5.1 *Assume that $\phi \in L^2(\mathbb{R})$ has support on an interval $[0, N]$ and that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence. Assume further that*

$$\sum_{k=0}^{N-1} c_k \phi(x+k) = 0,\ \forall x \in [0,1] \Rightarrow c_0 = 0. \tag{7}$$

Then $\{T_k\phi\}_{k\in\mathbb{Z}}$ has an oblique dual $\{T_k\tilde{\phi}\}_{k\in\mathbb{Z}}$ generated by a function of the form

$$\tilde{\phi}(x) = \left(a_0 + a_1 x + \cdots + a_M x^M\right) \chi_{[0,1]}(x) \tag{8}$$

for some $M \in \mathbb{N}$.

Proof. We have to prove that the moment problem (4) has a solution $\tilde{\phi}$ of the form (8) for some $M \in \mathbb{N}$. But since the polynomials are dense in $L^2(0,1)$, this follows directly from Theorem 2.3. $\square$

We note that the proof of Theorem 5.1 relies on Weierstrass' theorem and is non-constructive.

Exactly as in Theorem 4.1, the condition (7) in Theorem 5.1 is invariant under multiplication with functions $x^{p/2}(1-x)^{q/2}$. Thus, analog to earlier results, we have

Corollary 5.2 *Under the assumptions in Theorem 5.1, for any $p, q \in \mathbb{N}$ we can find am oblique dual generated by a function of the form*

$$\tilde{\phi}(x) = \left(a_0 + a_1 x + \cdots + a_M x^M\right) x^p (1-x)^q \chi_{[0,1]}(x)$$

for some $M \in \mathbb{N}$.

6 Appendix

In this section we provide some general information on moment problems, as well as proofs of Lemma 2.2 and Theorem 2.3.

Let $\mathcal{H}$ be a separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. Given a countable sequence $\{f_k\}_{k\in I}$ of elements in $\mathcal{H}$ and $\{a_k\}_{k\in I} \in \ell^2(I)$, we ask if we can find $f \in \mathcal{H}$ such that

$$\langle f, f_k \rangle = a_k, \ \forall k \in I. \tag{9}$$

A problem of this type is called a moment problem. It is clear that there are cases where no solution exist, and other cases where infinitely many solutions exist. If $\{f_k\}_{k\in I}$ is a Riesz sequence, then there exists a unique solution f belonging to $\overline{\text{span}}\{f_k\}_{k\in I}$.

Proof of Lemma 2.2. Assume first that (i) is satisfied, i.e., (2) has a solution f. Then, if $\sum_{k=1}^N c_k f_k = 0$ for some coefficients $\{c_k\}_{k=1}^N$, we have that

$$0 = \langle f, \sum_{k=1}^N c_k f_k \rangle = \sum_{k=1}^N c_k \langle f, f_k \rangle = c_1,$$

i.e., (ii) holds. Now assume that (ii) is satisfied. Then $f_1 \notin \text{span}\{f_k\}_{k=2}^N$. Let P denote the orthogonal projection of $\mathcal{H}$ onto $\text{span}\{f_k\}_{k=2}^N$, and put $\varphi = f_1 - Pf_1$. Then

$$\langle \varphi, f_1 \rangle = \langle f_1 - Pf_1, f_1 - Pf_1 \rangle + \langle f_1 - Pf_1, Pf_1 \rangle = ||f_1 - Pf_1||^2 \neq 0,$$

and $\langle \varphi, f_k \rangle = 0$ for $k = 2, \ldots, N$. Thus

$$f := \frac{\varphi}{||f_1 - Pf_1||^2}$$

solves the moment problem (2), i.e., (i) is satisfied. The equivalence of (ii) and (iii) is clear. By construction, $f \in \operatorname{span}\{f_k\}_{k=1}^N$. $\square$

We now want to prove Theorem 2.3. We begin with an elementary lemma. Let $\mathbb{R}^{\mathbb{N}}$ denote the vector space consisting of real scalar-valued sequences, indexed by $\mathbb{N}$.

Lemma 6.1 *Let $N \in \mathbb{N}$, and assume that the vectors*

$$v_1 = \begin{pmatrix} v_{11} \\ v_{12} \\ \cdot \\ \\ \cdot v_{1k} \\ \cdot \\ \cdot \end{pmatrix}, \quad v_2 = \begin{pmatrix} v_{21} \\ v_{22} \\ \cdot \\ \cdot \\ v_{2k} \\ \cdot \\ \cdot \end{pmatrix}, \cdots, \; v_N = \begin{pmatrix} v_{N1} \\ v_{N2} \\ \cdot \\ \cdot \\ v_{Nk} \\ \cdot \\ \cdot \end{pmatrix}$$

in $\mathbb{R}^{\mathbb{N}}$ are linearly independent. Then there exists $M \in \mathbb{N}$ such that the vectors

$$\begin{pmatrix} v_{11} \\ v_{12} \\ \cdot \\ \cdot \\ v_{1M} \end{pmatrix}, \quad \begin{pmatrix} v_{21} \\ v_{22} \\ \cdot \\ \cdot \\ v_{2M} \end{pmatrix}, \cdots, \begin{pmatrix} v_{N1} \\ v_{N2} \\ \cdot \\ \cdot \\ v_{NM} \end{pmatrix}$$

are linearly independent in $\mathbb{R}^M$.

Proof. Assume that $v_1, v_2, \dots, v_N$ are linearly independent in $\mathbb{R}^{\mathbb{N}}$. In order to arrive at a contradiction, assume that for each $M \in \mathbb{N}$, there exists a sequence $\{c_k^M\}_{k=1}^N \in \mathbb{R}^N \setminus \{0\}$ such that

$$\sum_{k=1}^N c_k^M \begin{pmatrix} v_{k1} \\ v_{k2} \\ \cdot \\ \cdot \\ v_{kM} \end{pmatrix} = 0.$$

We choose $\{c_k^M\}_{k=1}^N$ such that $\left|\{c_k^M\}_{k=1}^N\right| = 1$. Due to compactness of the unit ball in $\mathbb{R}^N$, we can find a subsequence (call again the members of the sequence $\{c_k^M\}_{k=1}^N$) which is convergent,

$$\{c_k^M\}_{k=1}^N \to \{c_k\}_{k=1}^N \text{ as } M \to \infty.$$

By construction, we conclude that

$$\sum_{k=1}^{N} c_k v_k = 0,$$

which is a contradiction. $\square$

Lemma 6.2 *Let $\{f_k\}_{k=1}^N$ and $\{g_k\}_{k=0}^\infty$ be sequences in $\mathcal{H}$; assume that $\{f_k\}_{k=1}^N$ is linearly independent and that $\{g_k\}_{k=0}^\infty$ is total in $\mathcal{H}$. Then there exists $M \in \mathbb{N}$ such that the moment problem* (2) *has a solution $f \in span\{g_k\}_{k=0}^M$.*

Proof. Let $M \in \mathbb{N}$. Any $f \in \text{span}\{g_k\}_{k=0}^M$ can be written on the form $f = \sum_{k=0}^M a_k g_k$; then the moment problem (2) takes the form

$$\begin{cases} 1 = \langle f_1, f\rangle = \sum_{k=0}^{M} a_k \langle f_1, g_k\rangle \\ 0 = \langle f_2, f\rangle = \sum_{k=0}^{M} a_k \langle f_2, g_k\rangle \\ 0 = \cdots\cdots \\ 0 = \langle f_N, f\rangle = \sum_{k=0}^{M} a_k \langle f_N, g_k\rangle. \end{cases} \tag{10}$$

On matrix form, the equations have the form

$$\begin{pmatrix} \langle f_1, g_0\rangle & \langle f_1, g_1\rangle & \cdot & \cdot & \langle f_1, g_M\rangle \\ \langle f_2, g_0\rangle & \langle f_2, g_1\rangle & \cdot & \cdot & \langle f_2, g_M\rangle \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \langle f_N, g_0\rangle & \langle f_N, g_1\rangle & \cdot & \cdot & \langle f_N, g_M\rangle \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ \cdots \\ a_M \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ \cdots \\ 0 \end{pmatrix} \tag{11}$$

Note that the vectors

$$v_1 = \begin{pmatrix} \langle f_1, g_0\rangle \\ \langle f_1, g_1\rangle \\ \cdot \\ \cdot \\ \langle f_1, g_M\rangle \\ \cdot \\ \cdot \end{pmatrix}, v_2 = \begin{pmatrix} \langle f_2, g_0\rangle \\ \langle f_2, g_1\rangle \\ \cdot \\ \cdot \\ \langle f_2, g_M\rangle \\ \cdot \\ \cdot \end{pmatrix}, \ldots, v_N = \begin{pmatrix} \langle f_N, g_0\rangle \\ \langle f_N, g_1\rangle \\ \cdot \\ \cdot \\ \langle f_N, g_M\rangle \\ \cdot \\ \cdot \end{pmatrix}$$

are linearly independent in $\mathbb{R}^{\mathbb{N}}$. In fact, if

$$\sum_{k=1}^{N} c_k v_k = 0,$$

then

$$\langle \sum_{k=1}^{N} c_k f_k, g_j \rangle = 0$$

for all $j = 0, 1, 2, ...$ Since $\text{span}\{g_j\}_{j=0}^{\infty}$ is dense in $\mathcal{H}$ this implies that

$$\sum_{k=1}^{N} c_k f_k = 0$$

and therefore, by assumption, that $c_k = 0$ for all k. According to Lemma 6.1, there exists $M \in \mathbb{N}$ such that the N vectors

$$\begin{pmatrix} \langle f_1, g_0 \rangle \\ \langle f_1, g_1 \rangle \\ \cdot \\ \cdot \\ \langle f_1, g_M \rangle \end{pmatrix}, \begin{pmatrix} \langle f_2, g_0 \rangle \\ \langle f_2, g_1 \rangle \\ \cdot \\ \cdot \\ \langle f_2, g_M \rangle \end{pmatrix}, \dots, \begin{pmatrix} \langle f_N, g_0 \rangle \\ \langle f_N, g_1 \rangle \\ \cdot \\ \cdot \\ \langle f_N, g_M \rangle \end{pmatrix}$$

are linearly independent. Thus, with this choice of M, the row-rank of the matrix in (11) is N; therefore also the rank of the column-space is N, i.e., the matrix is surjective, and the equation solvable. $\square$

Proof of Theorem 2.3:
Suppose that $f_1 \notin \text{span}\{f_2, \cdots, f_N\}$ and that $\{g_k\}_{k=0}^{\infty}$ is total. Then we can decompose $\{2, \cdots, N\} = A \cup B$ such that $\{f_1\} \cup \{f_i\}_{i \in A}$ is linearly independent and $\{f_i\}_{i \in B} \subset \text{span}\{f_i\}_{i \in A}$. By Lemma 6.2 there exist $M \in \mathbb{N}$ and $g \in \text{span}\{g_j\}_{j=1}^{M}$ such that $\langle f_1, g \rangle = 1$ and $\langle f_i, g \rangle = 0$ for $i \in A$. Then obviously, $\langle f_i, g \rangle = 0$ for $i \in B$. Hence the moment problem has a solution in $\text{span}\{g_j\}_{j=1}^{M}$. $\square$

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