# Pairs of explicitly given dual Gabor frames in $L^2(\mathbb{R}^d)$

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**Abstract**

Given certain compactly supported functions $g \in L^2(\mathbb{R}^d)$ whose $\mathbb{Z}^d$-translates form a partition of unity, and real invertible $d \times d$ matrices $B, C$ for which $||C^T B||$ is sufficiently small, we prove that the Gabor system $\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ forms a frame, with a (non-canonical) dual Gabor frame generated by an explicitly given finite linear combination of shifts of $g$. For functions $g$ of the above type and arbitrary real invertible $d \times d$ matrices $B, C$ this result leads to a construction of a multi–Gabor frame $\{E_{Bm}T_{Cn}g_k\}_{m,n\in\mathbb{Z}^d,k\in\mathcal{F}}$, where all the generators $g_k$ are dilated and translated versions of $g$. Again, the dual generators have a similar form, and are given explicitly. Our concrete examples concern box splines.

## 1 Introduction

For $y \in \mathbb{R}^d$, the translation operator $T_y$ and the modulation operator $E_y$ are defined by

$$\begin{aligned}(T_y f)(x) &= f(x-y), \quad x \in \mathbb{R}^d,\\ (E_y f)(x) &= e^{2\pi i y\cdot x} f(x), \quad x \in \mathbb{R}^d,\end{aligned}$$

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where $y \cdot x$ denotes the inner product between $y$ and $x$ in $\mathbb{R}^d$. Given two real and invertible $d \times d$ matrices $B$ and $C$ we consider Gabor systems of the form

$$\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d} = \{e^{2\pi i Bm\cdot x}g(x - Cn)\}_{m,n\in\mathbb{Z}^d}.$$

Our purpose is to construct a class of Gabor frames with generators that are easy to use in practice, and having the additional property that we can find a dual generator of the form

$$h = \sum_{k\in\mathcal{F}} c_k T_k g$$

for some finite set $\mathcal{F} \subset \mathbb{Z}^d$ and explicitly given scalar coefficients $c_k$. One advantage of this is that the decay of the dual generator $h$ in the frequency domain is controlled by the decay of $\hat{g}$. Our results extend the one-dimensional results in [2]. As we will see, the extension is non-trivial: it is not clear from the one-dimensional version how one has to define the dual generators in higher dimensions.

Our approach is strongly connected with the results by Janssen [5], [6], Labate [7], Hernandez, Labate and Weiss [4], and Ron and Shen [8],[9]. However, in contrast to these papers, the focus is on explicit constructions rather than general characterizations. For more information about Gabor systems and their role in time-frequency analysis we refer to the book [3] by Gröchenig; for general frame theory we refer to [1].

In the rest of the introduction we collect a few conventions about notation and a basic result for obtaining a pair of dual frames. The dilation operator associated with a real $d \times d$ matrix $C$ is

$$(D_C f)(x) = |\det C|^{1/2} f(Cx), \quad x \in \mathbb{R}^d.$$

Let $C^T$ denote the transpose of a matrix $C$; then

$$D_C E_y = E_{C^T y} D_C, \quad D_C T_y = T_{C^{-1}y} D_C.$$

If $C$ is invertible, we use the notation

$$C^\sharp = (C^T)^{-1}.$$

For $f \in (L^1 \cap L^2)(\mathbb{R}^d)$ we denote the Fourier transform by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\gamma} dx.$$

As usual, the Fourier transform is extended to a unitary operator on $L^2(\mathbb{R}^d)$. The reader can check that

$$\mathcal{F}T_{Ck} = E_{-Ck}\mathcal{F}.$$

We conclude the introduction by stating a special case of a result from [4]; it will form the basis for all the results presented in the paper. Let

$$\mathcal{D} := \{f \in L^2(\mathbb{R}^d) : \hat{f} \in L^\infty(\mathbb{R}^d) \text{ and supp}\hat{f} \text{ is compact}\}.$$

**Lemma 1.1** *Let $B$ be an invertible $d \times d$ matrix, and let $\{g_n\}_{n\in\mathbb{Z}^d}$ and $\{h_n\}_{n\in\mathbb{Z}^d}$ be collections of functions in $L^2(\mathbb{R}^d)$. Assume that $\{T_{Bm}g_n\}_{m,n\in\mathbb{Z}^d}$ and $\{T_{Bm}h_n\}_{m,n\in\mathbb{Z}^d}$ are Bessel sequences and that for all $f \in \mathcal{D}$,*

$$\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}\int_{supp\hat{f}} |\hat{f}(\gamma + B^\sharp m)|^2 |\hat{g}_n(\gamma)|^2 d\gamma \;<\; \infty, \tag{1}$$

$$\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}\int_{supp\hat{f}} |\hat{f}(\gamma + B^\sharp m)|^2 |\hat{h}_n(\gamma)|^2 d\gamma \;<\; \infty. \tag{2}$$

*Then $\{T_{Bm}g_n\}_{m,n\in\mathbb{Z}^d}$ and $\{T_{Bm}h_n\}_{m,n\in\mathbb{Z}^d}$ are dual frames for $L^2(\mathbb{R}^d)$ if and only if*

$$\sum_{k\in\mathbb{Z}^d} \overline{\hat{g}_k(\gamma - B^\sharp n)}\hat{h}_k(\gamma) = |\det B|\delta_{n,0}, \;\; a.e.\gamma,$$

*for all $n \in \mathbb{Z}^d$.*

## 2 Dual pairs of Gabor frames

We first prove a time-domain version of Lemma 1.1 for Gabor systems. As we will see, we can remove the technical conditions (1) and (2) in the Gabor case. We begin with a Lemma.

**Lemma 2.1** *Let $g \in L^2(\mathbb{R}^d)$ and assume that $B$ and $C$ are invertible matrices. Then for all $f \in \mathcal{D}$,*

$$\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}\int_{supp\hat{f}} |\hat{f}(\gamma + B^\sharp m)|^2 |g(\gamma - Cn)|^2 d\gamma < \infty.$$

**Proof.** Let $f \in \mathcal{D}$. Then

$$\sum_{m\in\mathbb{Z}^d} |\hat{f}(\gamma + B^\sharp m)|^2 \quad \leq \quad \sup_{\gamma\in B^\sharp[0,1]^d} \sum_{m\in\mathbb{Z}^d} |\hat{f}(\gamma + B^\sharp m)|^2. \tag{3}$$

Independently of the choice of $\gamma \in B^\sharp[0,1]^d$, only a fixed finite number of $m \in \mathbb{Z}^d$ will give non-zero contributions to the sum on the right-hand side of (3); since $\hat{f}$ is bounded, this implies that there exists a constant $K$ such that

$$\sum_{m\in\mathbb{Z}^d} |\hat{f}(\gamma + B^\sharp m)|^2 \leq K, \ \ a.e \ \gamma.$$

Hence,

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} \int_{\mathrm{supp}\hat{f}} |\hat{f}(\gamma + B^\sharp m)|^2 |g(\gamma - Cn)|^2 d\gamma \\
&= \int_{\mathrm{supp}\hat{f}} \sum_{m\in\mathbb{Z}^d} |\hat{f}(\gamma + B^\sharp m)|^2 \sum_{n\in\mathbb{Z}^d} |g(\gamma - Cn)|^2 d\gamma \\
&\leq K \int_{\mathrm{supp}\hat{f}} \sum_{n\in\mathbb{Z}^d} |g(\gamma - Cn)|^2 d\gamma.
\end{aligned}$$

Choose an integer $a > 0$ such that

$$\mathrm{supp}\hat{f} \subseteq C[-a,a]^d.$$

Then

$$\begin{aligned}
\int_{\mathrm{supp}\hat{f}} \sum_{n\in\mathbb{Z}^d} |g(\gamma - Cn)|^2 d\gamma \quad &\leq \int_{C[-a,a]^d} \sum_{n\in\mathbb{Z}^d} |g(\gamma - Cn)|^2 d\gamma \\
&\leq |\det C| \int_{[-a,a]^d} \sum_{n\in\mathbb{Z}^d} |g(C(\xi - n))|^2 d\xi.
\end{aligned}$$

Now, using that (modulo null-sets)

$$[-a,a]^d = \bigcup_{k\in[-a,a-1]^d\cap\mathbb{Z}^d} (k + [0,1]^d)$$

and that the function $\xi \mapsto \sum_{n\in\mathbb{Z}^d} |g(C(\xi-n))|^2$ is $\mathbb{Z}^d$-periodic,

$$\begin{aligned}
&\int_{[-a,a]^d} \sum_{n\in\mathbb{Z}^d} |g(C(\xi-n))|^2 d\xi \\
&= (2a)^d \int_{[0,1]^d} \sum_{n\in\mathbb{Z}^d} |g(C(\xi-n))|^2 d\xi \\
&= (2a)^d \int_{\mathbb{R}^d} |g(C\xi)|^2 d\xi \\
&= |\det C|^{-1} (2a)^d \int_{\mathbb{R}^d} |g(\eta)|^2 d\eta < \infty.
\end{aligned}$$

□

The following is the frame-pair version of Corollary 3.3 in [7]. It can also be considered as the time-domain version of Lemma 1.1. Results of that type already appeared in [8] by Ron and Shen, and (in the one-dimensional case) in [5] by Janssen. We provide the short proof for the sake of completeness.

**Lemma 2.2** *Two Bessel sequences $\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ and $\{E_{Bm}T_{Cn}h\}_{m,n\in\mathbb{Z}^d}$ form dual frames for $L^2(\mathbb{R}^d)$ if and only if*

$$\sum_{k\in\mathbb{Z}^d} \overline{g(x - B^\sharp n - Ck)}h(x - Ck) = |\det B|\delta_{n,0}. \tag{4}$$

**Proof.** We note that $\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ and $\{E_{Bm}T_{Cn}h\}_{m,n\in\mathbb{Z}^d}$ form dual frames if and only if $\{\mathcal{F}^{-1}E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ and $\{\mathcal{F}^{-1}E_{Bm}T_{Cn}h\}_{m,n\in\mathbb{Z}^d}$ are dual frames. Now, $\mathcal{F}^{-1}E_{Bm}T_{Cn}g = T_{-Bm}\mathcal{F}^{-1}T_{Cn}g$; thus, the result follows from Lemma 1.1 and Lemma 2.1 with $g_n = \mathcal{F}^{-1}T_{Cn}g, h_n = \mathcal{F}^{-1}T_{Cn}h$. □

We now present the first version of our results. For simplicity we consider the case $C = I$. For any $d \times d$ matrix we define the norm $||B||$ by

$$||B|| = \sup_{||x||=1} ||Bx||.$$

**Theorem 2.3** *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R}^d)$ be a real-valued bounded function with supp $g \subseteq [0, N]^d$, for which*

$$\sum_{n\in\mathbb{Z}^d} g(x - n) = 1.$$

*Assume that the $d \times d$ matrix $B$ is invertible and $||B|| \leq \frac{1}{\sqrt{d}(2N-1)}$. For $i = 1, \ldots, d$, let $F_i$ be the set of lattice points $\{k_j\}_{j=1}^d \in \mathbb{Z}^d$ for which the coordinates $k_j, j = 1, \ldots, d$, satisfy the requirements*

$$\begin{cases} \text{if } j = 1, \ldots, i-1, \text{ then } |k_j| \leq N-1; \\ \text{if } j = i, \quad \text{then } 1 \leq k_j \leq N-1; \\ \text{if } j = i+1, \ldots, d, \text{ then } k_j = 0. \end{cases} \tag{5}$$

*Define $h \in L^2(\mathbb{R}^d)$ by*

$$h(x) := |\det B| \left[ g(x) + 2 \sum_{i=1}^{d} \sum_{k \in F_i} g(x+k) \right]. \tag{6}$$

*Then the function $g$ and the function $h$ generate dual frames $\{E_{Bm}T_n g\}_{m,n \in \mathbb{Z}^d}$ and $\{E_{Bm}T_n h\}_{m,n \in \mathbb{Z}^d}$ for $L^2(\mathbb{R}^d)$.*

**Proof.** We apply Lemma 2.2. Since $B$ is invertible, for any $n \in \mathbb{Z}^d$ we have

$$|n| = ||B^T B^\sharp n|| \leq ||B|| \, ||B^\sharp n||;$$

thus, for $n \neq 0$, $||B^\sharp n|| \geq 1/||B||$. Note that with the definition (6), we have $\operatorname{supp} h \subseteq [-N+1, 2N-1]^d$; thus (4) is satisfied for $n \neq 0$ if $1/||B|| \geq \sqrt{d}(2N-1)$, i.e., if

$$||B|| \leq \frac{1}{\sqrt{d}(2N-1)}.$$

Thus, we only need to check that

$$\sum_{k \in \mathbb{Z}^d} g(x-k)h(x-k) = |\det B|, \ x \in [0,1]^d;$$

due to the compact support of $g$, this is equivalent to

$$\sum_{n \in [0,N-1]^d \cap \mathbb{Z}^d} g(x+n)h(x+n) = |\det B|, \ x \in [0,1]^d. \tag{7}$$

To check that (7) holds, we use that for $x \in [0,1]^d$,

$$\sum_{n \in [0,N-1]^d \cap \mathbb{Z}^d} g(x+n) = 1. \tag{8}$$

For $n := \{n_j\}_{j=1}^d \in [0, N-1]^d \cap \mathbb{Z}^d$, and $i = 1, \ldots, d$, let $E_i^n$ denote the set of lattice points $\{k_j\}_{j=1}^d \in \mathbb{Z}^d$ whose coordinates $k_j$ satisfy the requirements

$$\begin{cases} \text{if } j = 1, \ldots, i-1, \text{ then } 0 \le k_j \le N-1; \\ \text{if } j = i, \text{ then } n_j + 1 \le k_j \le N-1; \\ \text{if } j = i+1, \ldots, d, \text{ then } k_j = n_j. \end{cases}$$

Define $\tilde{h}_n \in L^2(\mathbb{R}^d)$ by

$$\tilde{h}_n(x) \quad := \quad |\det B| \left[ g(x+n) + 2 \sum_{i=1}^{d} \sum_{k \in E_i^n} g(x+k) \right].$$

We now consider the finite set $[0, N-1]^d \cap \mathbb{Z}^d$. Using lexicographic ordering, *i.e.*

$$\begin{aligned} &(i_1, \ldots, i_d) > (j_1, \ldots, j_d) \\ &\Leftrightarrow (i_d > j_d) \vee ((i_d = j_d) \wedge (i_{d-1} > j_{d-1})) \vee \cdots \\ &\vee((i_d = j_d) \wedge \cdots \wedge (i_2 = j_2) \wedge i_1 > j_1), \end{aligned}$$

we write

$$[0, N-1]^d \cap \mathbb{Z}^d = \{n_1, n_2, \cdots, n_{N^d}\},$$

with $n_j < n_k$ for $j < k$. Then for $x \in [0,1]^d$, (8) implies that

$$\begin{aligned} 1 &= \left( \sum_{j=1}^{N^d} g(x+n_j) \right)^2 \\ &= (g(x+n_1) + g(x+n_2) + \cdots + g(x+n_{N^d})) \times \\ &\qquad (g(x+n_1) + g(x+n_2) + \cdots + g(x+n_{N^d})) \\ &= g(x+n_1)[g(x+n_1) + 2g(x+n_2) + 2g(x+n_3) + \cdots + 2g(x+n_{N^d})] \\ &\quad + g(x+n_2)[g(x+n_2) + 2g(x+n_3) + 2g(x+n_4) + \cdots + 2g(x+n_{N^d})] \\ &\quad + \cdots \\ &\quad + \cdots \\ &\quad + g(x+n_{N^d-1})[g(x+n_{N^d-1}) + 2g(x+n_{N^d})] \\ &\quad + g(x+n_{N^d})[g(x+n_{N^d})] \\ &= \frac{1}{|\det B|} \sum_{j=1}^{N^d} g(x+n_j) \tilde{h}_{n_j}(x). \end{aligned}$$

It remains to show that for $x \in [0,1]^d$ and $n = \{n_j\}_{j=1}^d \in [0, N-1]^d \cap \mathbb{Z}^d$,

$$h(x+n) = \tilde{h}_n(x).$$

In order to do so, it is sufficient to show that for any $i = 1, \dots, d$,

$$\sum_{k \in F_i} g(x+n+k) = \sum_{k \in E_i^n} g(x+k), \;\; x \in [0,1]^d. \tag{9}$$

Fix $i \in \{1, \dots, d\}$. If $1 \le j < i$, then

$$\begin{aligned} \{n_j + k_j : \{k_j\}_{j=1}^d \in F_i\} &= [n_j - N + 1, n_j + N - 1] \cap \mathbb{Z} \qquad (10) \\ &\supseteq [0, N-1] \cap \mathbb{Z}. \end{aligned}$$

If $j = i$, then

$$\begin{aligned} \{n_j + k_j : \{k_j\}_{j=1}^d \in F_i\} &= [n_j + 1, n_j + N - 1] \cap \mathbb{Z} \qquad (11) \\ &\supseteq [1 + n_j, N-1] \cap \mathbb{Z}. \end{aligned}$$

If $j > i$, then

$$\{n_j + k_j : \{k_j\}_{j=1}^d \in F_i\} = \{n_j\}.$$

Via the definition of the set $E_i^n$ this shows that

$$E_i^n \subseteq \{n + k : k = \{k_j\}_{j=1}^d \in F_i\}. \tag{12}$$

In order to show that we have equality in (9), we again fix $i \in \{1, \dots, d\}$. Suppose that $m := \{m_j\}_{j=1}^d \in \{n + k : k = \{k_j\}_{j=1}^d \in F_i\} \setminus E_i^n$. Then either, by (10), there exists $j \in \{1, \dots, i-1\}$ such that

$$m_j := n_j + k_j \notin [0, N-1] \cap \mathbb{Z};$$

or, by (11),

$$m_i := n_i + k_i \in ([1 + n_j, n_j + N - 1] \setminus [1 + n_j, N-1]) \cap \mathbb{Z} = [N, n_j + N - 1] \cap \mathbb{Z}.$$

In both cases, since $\mathrm{supp} g \subseteq [0, N]^d$, this implies that $g(x+m) = 0$ for $x \in [0,1]^d$. Hence,

$$\sum_{k \in F_i} g(x+n+k) = \sum_{k \in E_i^n} g(x+k),$$

as desired. $\square$

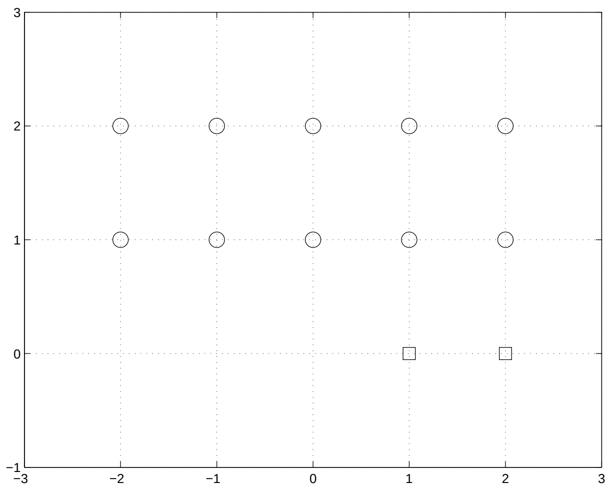


Figure 1: The sets $F_1$ (marked by $\Box$) and $F_2$ (marked by $\bigcirc$) corresponding to $N = 3$ and $d = 2$.

**Example 2.4** For $d = 1$, the Gabor system considered in Theorem 2.3 is $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ for some $b > 0$. The reader can check that

$$F_1 = \{1, \dots, N-1\};$$

thus, the expression for the dual generator $h$ in (6) is

$$h(x) = bg(x) + 2b \sum_{k=1}^{N-1} g(x+k).$$

This result corresponds to the one-dimensional case treated in [2].

For $d = 2$, (5) leads to the sets

$$\begin{aligned} F_1 &= \{(k_1, k_2) \in \mathbb{Z}^2 |\ 1 \le k_1 \le N-1, k_2 = 0\}, \\ F_2 &= \{(k_1, k_2) \in \mathbb{Z}^2 |\ |k_1| \le N-1, 1 \le k_2 \le N-1\}. \end{aligned}$$

For $N = 3$, the sets $F_1$ and $F_2$ are marked on Figure 1.

Via a change of variable Theorem 2.3 leads to a construction of frames of the type $\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ and convenient duals:

**Theorem 2.5** *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R}^d)$ be a real-valued bounded function with supp $g \subseteq [0, N]^d$, for which*

$$\sum_{n\in\mathbb{Z}^d} g(x-n) = 1.$$

*Let $B$ and $C$ be invertible $d \times d$ matrices such that $||C^T B|| \leq \frac{1}{\sqrt{d}(2N-1)}$, and let (with the sets $F_i$ defined as in Theorem 2.3)*

$$h(x) = |\det(C^T B)| \left[ g(x) + 2 \sum_{i=1}^{d} \sum_{k\in F_i} g(x+k) \right]. \tag{13}$$

*Then the function $D_{C^{-1}}g$ and the function $D_{C^{-1}}h$ generate dual Gabor frames $\{E_{Bm}T_{Cn}D_{C^{-1}}g\}_{m,n\in\mathbb{Z}^d}$ and $\{E_{Bm}T_{Cn}D_{C^{-1}}h\}_{m,n\in\mathbb{Z}^d}$ for $L^2(\mathbb{R}^d)$.*

**Proof.** By assumptions and Theorem 2.3, the Gabor systems $\{E_{C^T Bm}T_n g\}_{m,n\in\mathbb{Z}^d}$ and $\{E_{C^T Bm}T_n h\}_{m,n\in\mathbb{Z}^d}$ form dual frames; since

$$D_{C^{-1}} E_{C^T Bm} T_n = E_{Bm} T_{Cn} D_{C^{-1}},$$

the result follows from $D_{C^{-1}}$ being unitary. $\square$

For functions $g$ of the above type and arbitrary real invertible $d \times d$ matrices $B$ and $C$, Theorem 2.5 leads to a construction of a (finitely generated) multi–Gabor frame $\{E_{Bm}T_{Cn}g_k\}_{m,n\in\mathbb{Z}^d,k\in\mathcal{F}}$, where all the generators $g_k$ are dilated and translated versions of $g$. Again, the dual generators have a similar form, and are given explicitly:

**Theorem 2.6** *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R}^d)$ be a real-valued bounded function with supp $g \subseteq [0, N]^d$, for which*

$$\sum_{n\in\mathbb{Z}^d} g(x-n) = 1.$$

*Let $B$ and $C$ be invertible $d \times d$ matrices and choose $J \in \mathbb{N}$ such that $J \geq ||C^T B|| \sqrt{d}(2N-1)$. Define the function $h$ by (13). Then the functions*

$$g_k = T_{\frac{1}{J}Ck} D_{JC^{-1}} g, \quad h_k = T_{\frac{1}{J}Ck} D_{JC^{-1}} h, \; k \in \mathbb{Z}^d \cap [0, J-1]^d$$

*generate dual multi-Gabor frames $\{E_{Bm}T_{Cn}g_k\}_{m,n\in\mathbb{Z}^d,k\in\mathbb{Z}^d\cap[0,J-1]^d}$ and $\{E_{Bm}T_{Cn}h_k\}_{m,n\in\mathbb{Z}^d,k\in\mathbb{Z}^d\cap[0,J-1]^d}$ for $L^2(\mathbb{R}^d)$.*

**Proof.** The choice of $J$ implies that the matrices $B$ and $\frac{1}{J}C$ satisfy the conditions in Theorem 2.5; thus

$$\{e^{2\pi iBm\cdot x}(D_{JC^{-1}}g)(x-\frac{1}{J}Cn)\}_{m,n\in\mathbb{Z}^d} \text{ and } \{e^{2\pi iBm\cdot x}(D_{JC^{-1}}h)(x-\frac{1}{J}Cn)\}_{m,n\in\mathbb{Z}^d}$$

form a pair of dual Gabor frames for $L^2(\mathbb{R}^d)$. Now,

$$\left\{\frac{1}{J}Cn\right\}_{n\in\mathbb{Z}^d} = \bigcup_{k\in\mathbb{Z}^d\cap[0,J-1]^d}\left\{\frac{1}{J}Ck+Cn\right\}_{n\in\mathbb{Z}^d}.$$

Thus

$$\begin{aligned}\left\{(D_{JC^{-1}}g)(\cdot-\frac{1}{J}Cn)\right\}_{n\in\mathbb{Z}^d} &= \bigcup_{k\in\mathbb{Z}^d\cap[0,J-1]^d}\left\{(D_{JC^{-1}}g)(\cdot-\frac{1}{J}Ck-Cn)\right\}_{n\in\mathbb{Z}^d}\\ &= \bigcup_{k\in\mathbb{Z}^d\cap[0,J-1]^d}\left\{T_{Cn}T_{\frac{1}{J}Ck}D_{JC^{-1}}g(\cdot)\right\}_{n\in\mathbb{Z}^d}.\end{aligned}$$

Inserting this into the expression for the pair of dual frames leads to the result. □

Note that multi-generated Gabor system have appeared in various applications for a long time, see, e.g., [10].

Via our results we now construct Gabor frames for $L^2(\mathbb{R}^d)$ with box spline generators and dual generators having a similar form.

**Example 2.7** Let $B_2$ be the one-dimensional $B$-spline of order 2 defined by

$$B_2(x) = \begin{cases} x, & x\in[0,1[;\\ 2-x, & x\in[1,2[;\\ 0, & x\notin[0,2[.\end{cases}$$

Define $g\in L^2(\mathbb{R}^2)$ by

$$g(x,y) = B_2(x)\ B_2(y); \tag{14}$$

then supp$g\subseteq[0,2]^2$, and

$$\sum_{n\in\mathbb{Z}^2} g(x-n) = 1,\ x\in\mathbb{R}^2,$$

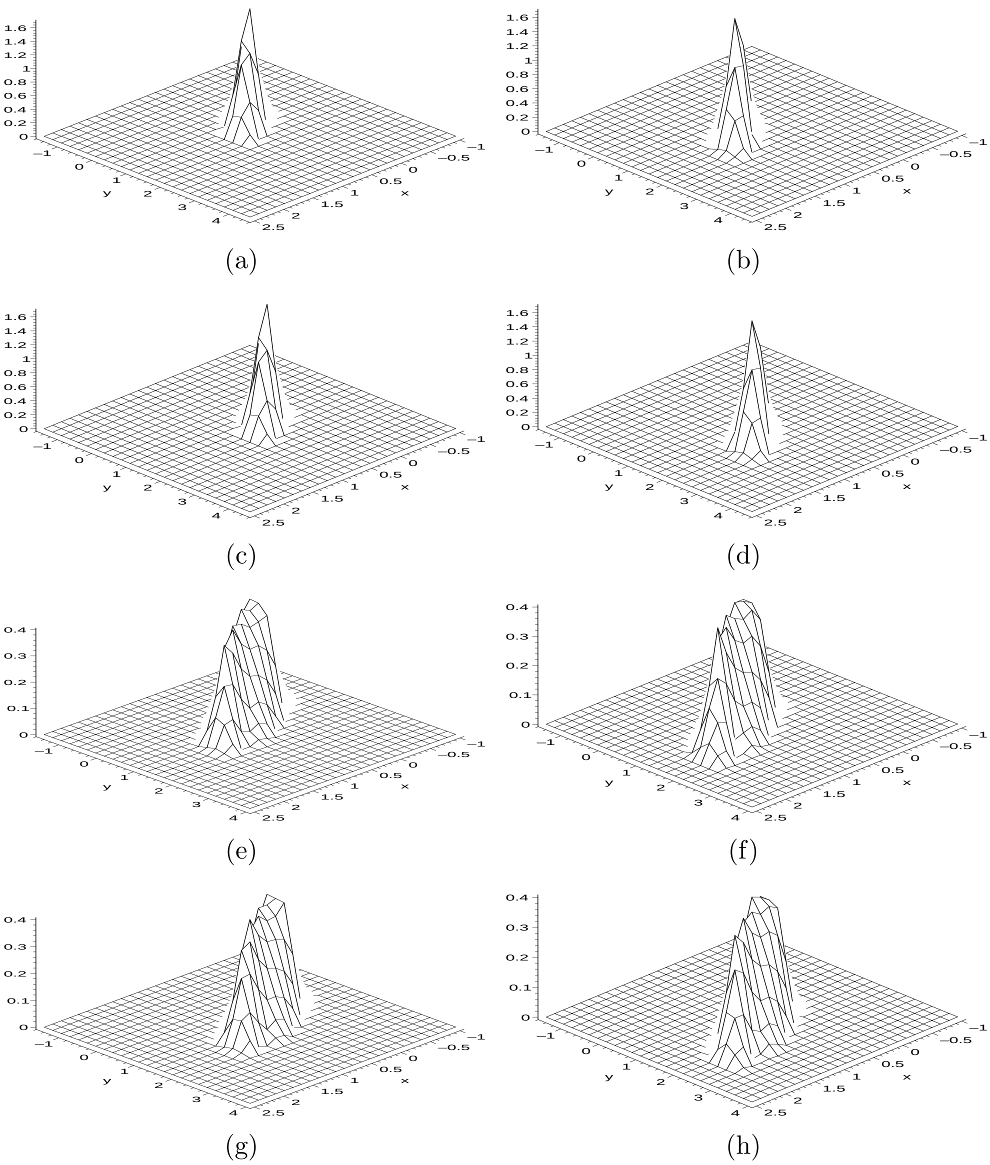

Figure 2: Plots of the generators in Example 2.7: (a) $g_{(0,0)}$; (b) $g_{(1,0)}$; (c) $g_{(0,1)}$; (d) $g_{(1,1)}$; (e) $h_{(0,0)}$; (f) $h_{(1,0)}$; (g) $h_{(0,1)}$; (h) $h_{(1,1)}$.

since the integer-translates of $B_2$ form a partition of unity. Let the $2 \times 2$ matrices $B$ and $C$ be defined by

$$B = \frac{1}{10}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

A direct calculation shows that

$$\begin{aligned} ||C^T B||^2 &= \left|\left|\frac{1}{10}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right|\right|^2 = \sup_{\theta} \left|\left|\frac{1}{10}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right|\right|^2 \\ &= \left(\frac{1}{10}\right)^2 (\sqrt{2}+1)^2. \end{aligned}$$

Thus

$$||C^T B||\sqrt{d}(2N-1) = \frac{3}{10}(2+\sqrt{2}) = 1.02\cdots.$$

Thus we can apply Theorem 2.6 with $J = 2$. Define the function $h \in L^2(\mathbb{R}^2)$ by (13), i.e.,

$$\begin{aligned} h(x,y) &= |\det(C^T B)|[g(x,y) + 2g((x,y)+(1,0)) \\ &+ 2g((x,y)+(-1,1)) + 2g((x,y)+(0,1)) + 2g((x,y)+(1,1))] \\ &= \frac{1}{10}\begin{cases} 2xy+2x+2y+2, & (x,y) \in [-1,0[\times[-1,0[; \\ 2x+2, & (x,y) \in [-1,0[\times[0,1[; \\ 4x-2xy+4-2y, & (x,y) \in [-1,0[\times[1,2[; \\ 2y+2, & (x,y) \in [0,1[\times[-1,0[; \\ -xy+2, & (x,y) \in [0,1[\times[0,1[; \\ -2x+xy+4-2y, & (x,y) \in [0,1[\times[1,2[; \\ 2y+2, & (x,y) \in [1,2[\times[-1,0[; \\ -xy+2, & (x,y) \in [1,2[\times[0,1[; \\ -2x+xy+4-2y, & (x,y) \in [1,2[\times[1,2[; \\ 6y+6-2xy-2x, & (x,y) \in [2,3[\times[-1,0[; \\ 6-6y-2x+2xy, & (x,y) \in [2,3[\times[0,1[; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{15}$$

By Theorem 2.6, the four functions

$$g_k = T_{\frac{1}{2}Ck} D_{2C^{-1}} g, \; k \in \mathbb{Z}^2 \cap [0,1]^2 \tag{16}$$

generate a multi-Gabor frame $\{E_{Bm}T_{Cn}g_k\}_{m,n\in\mathbb{Z}^2, k\in\mathbb{Z}^2\cap[0,1]^2}$, with a dual frame $\{E_{Bm}T_{Cn}h_k\}_{m,n\in\mathbb{Z}^2, k\in\mathbb{Z}^2\cap[0,1]^2}$, where

$$h_k = T_{\frac{1}{2}Ck} D_{2C^{-1}} h, \; k \in \mathbb{Z}^2 \cap [0,1]^2. \tag{17}$$

**Example 2.8** Similar calculations can be performed for any tensor product of B-splines. On Figure 3 we plot the box spline $g(x,y) = B_3(x)B_3(y)$ and the function $h$ in (13) for the choice

$$B = \frac{1}{10}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

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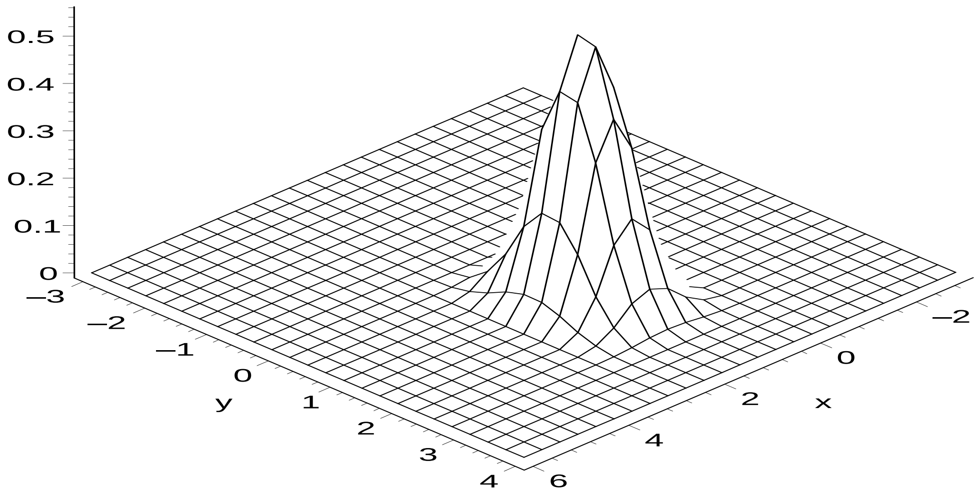
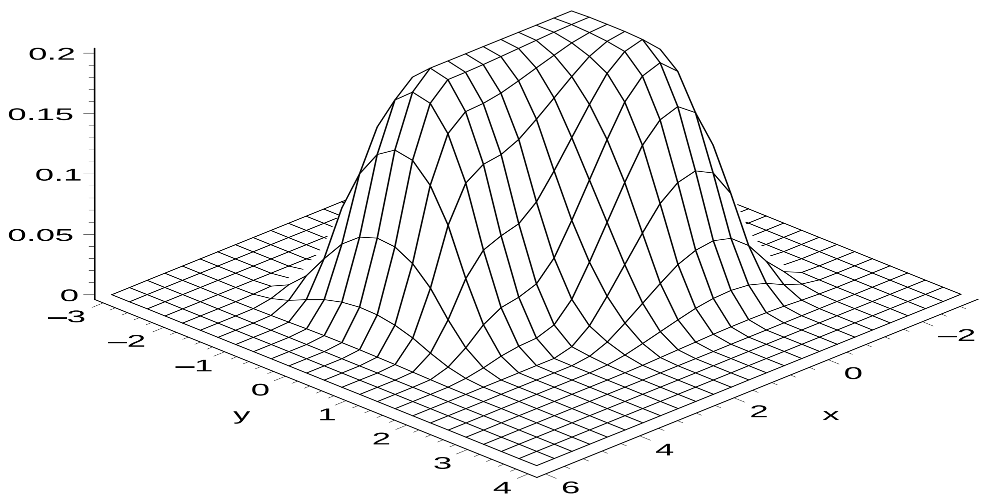



(a)




(b)

Figure 3: The functions $g$ (Figure (a)) and $h$ (Figure (b)) in Example 2.8.

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